Variational resolution for some general classes of nonlinear evolutions. Part I

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Abstract

We develop a variational technique for some wide classes of nonlinear evolutions. The novelty here is that we derive the main information directly from the corresponding Euler-Lagrange equations. In particular, we prove that not only the minimizer of the appropriate energy functional but also any critical point must be a solution of the corresponding evolutional system.

1 Introduction

Let X be a reflexive Banach space. Consider the following evolutional initial value problem:

$$\begin{cases} \frac{d}{dt}\big\{I\cdot u(t)\big\}+\Lambda_t\big(u(t)\big)=0 & \text{in } (0,T_0),\\ I\cdot u(0)=v_0. \end{cases} \tag{1.1}$$

Here $I:X\to X^*$ (X^* is the space dual to X) is a fixed bounded linear inclusion operator, which we assume to be self-adjoint and strictly positive, $u(t)\in L^q\big((0,T_0);X\big)$ is an unknown function, such that $I\cdot u(t)\in W^{1,p}\big((0,T_0);X^*\big)$ (where $I\cdot h\in X^*$ is the value of the operator I at the point $h\in X$), $\Lambda_t(x):X\to X^*$ is a fixed nonlinear mapping, considered for every fixed $t\in(0,T_0)$, and $v_0\in X^*$ is a fixed initial value. The most trivial variational principle related to (1.1) is the following one. Consider some convex function $\Gamma(y):X^*\to[0,+\infty)$, such that $\Gamma(y)=0$ if and only if $y=0$. Next define the following energy functional

$$E_0\big(u(\cdot)\big):=\int_0^{T_0}\Gamma\bigg(\frac{d}{dt}\big\{I\cdot u(t)\big\}+\Lambda_t\big(u(t)\big)\bigg)dt$$
$$\forall\, u(t)\in L^q\big((0,T_0);X\big)\ \text{ s.t. }\ I\cdot u(t)\in W^{1,p}\big((0,T_0);X^*\big)\ \text{ and }\ I\cdot u(0)=v_0\,. \tag{1.2}$$

Then it is obvious that $u(t)$ will be a solution to (1.1) if and only if $E_0\big(u(\cdot)\big)=0$. Moreover, the solution to (1.1) will exist if and only if there exists a minimizer $u_0(t)$ of the energy $E_0(\cdot)$, which satisfies $E_0\big(u_0(\cdot)\big)=0$.

We have the following generalization of this variational principle. Let $\Psi_t(x):X\to[0,+\infty)$ be some convex Gateux differentiable function, considered for every fixed $t\in(0,T_0)$ and such that $\Psi_t(0)=0$. Next define the Legendre transform of Ψ_t by

$$\Psi_t^*(y):=\sup\Big\{\big\langle z,y\big\rangle_{X\times X^*}-\Psi_t(z):\ z\in X\Big\}\qquad\forall y\in X^*\,. \tag{1.3}$$

It is well known that $\Psi_t^*(y):X^*\to\mathbb{R}$ is a convex function and

$$\Psi_t(x)+\Psi_t^*(y)\ \geq\ \big\langle x,y\big\rangle_{X\times X^*}\qquad\forall\, x\in X,\, y\in X^*\,, \tag{1.4}$$

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with equality if and only if $y = D\Psi_t(x)$. Next for $\lambda \in \{0,1\}$ define the energy

$$E_\lambda\big(u\big) := \int\limits_0^{T_0} \left\{\Psi_t\Big(\lambda u(t)\Big)+\Psi^*_t\Big(-\frac{d}{dt}\big\{I\cdot u(t)\big\}-\Lambda_t\big(u(t)\big)\Big)+\lambda\Big\langle u(t),\frac{d}{dt}\big\{I\cdot u(t)\big\}+\Lambda_t\big(u(t)\big)\Big\rangle_{X\times X^*}\right\}dt$$
$$\forall\, u(t) \in L^q\big((0,T_0);X\big) \ \text{ s.t. } \ I\cdot u(t) \in W^{1,p}\big((0,T_0);X^*\big) \ \text{ and } \ I\cdot u(0) = v_0. \quad (1.5)$$

Then, by (1.4) we have $E_\lambda(\,\cdot\,) \geq 0$ and moreover, $E_\lambda\big(u(\cdot)\big) = 0$ if and only if $u(t)$ is a solution to

$$\begin{cases} \frac{d}{dt}\big\{I\cdot u(t)\big\} + \Lambda_t\big(u(t)\big) + D\Psi_t\big(\lambda u(t)\big) = 0 \qquad \text{in } \ (0,T_0),\\ I\cdot u(0) = v_0 \end{cases} \tag{1.6}$$

(note here that since $\Psi_t(0) = 0$, in the case $\lambda = 0$ (1.6) coincides with (1.1). Moreover, if $\lambda = 0$ then the energy defined in (1.2) is a particular case of the energy in (1.5), where we take $\Gamma(x) := \Psi^*(-x)$). So, as before, a solution to (1.6) exists if and only if there exists a minimizer $u_0(t)$ of the energy $E_\lambda(\cdot)$, which satisfies $E_\lambda\big(u_0(\cdot)\big) = 0$. Consequently, in order to establish the existence of solution to (1.6) we need to answer the following questions:

(a) Does a minimizer to the energy in (1.5) exist?

(b) Does the minimizer $u_0(t)$ of the corresponding energy $E_\lambda(\cdot)$ satisfies $E_\lambda\big(u_0(\cdot)\big) = 0$?

To the best of our knowledge, the energy in (1.5) with $\lambda = 1$, related to (1.6), was first considered for the heat equation and other types of evolutions by Brezis and Ekeland in [1]. In that work they also first asked question **(b)**: If we don't know a priori that a solution of the equation (1.6) exists, how to prove that the minimum of the corresponding energy is zero. This question was asked even for very simple PDE's like the heat equation. A detailed investigation of the energy of type (1.5), with $\lambda = 1$, was done in a series of works of N. Ghoussoub and his coauthors, see the book [7] and also [8], [9], [10], [11]. In these works they considered a similar variational principle, not only for evolutions but also for some other classes of equations. They proved some theoretical results about general self-dual variational principles, which in many cases, can provide with the existence of a zero energy state (answering questions **(a)**+**(b)** together) and, consequently, with the existence of solution for the related equations (see [7] for details).

In this work we provide an alternative approach to the questions **(a)** and **(b)**. We treat them separately and in particular, for question **(b)**, we derive the main information by studying the Euler-Lagrange equations for the corresponding energy. To our knowledge, such an approach was first considered in [14] and provided there an alternative proof of existence of solution for initial value problems for some parabolic systems. Generalizing these results, we provide here the answer to questions **(a)** and **(b)** for some wide classes of evolutions. In particular, regarding question **(b)**, we are able to prove that in some general cases not only the minimizer but also any critical point $u_0(t)$ (i.e. any solution of corresponding Euler-Lagrange equation) satisfies $E_\lambda\big(u_0(\cdot)\big) = 0$, i.e. is a solution to (1.6).

The approach of Ghoussoub in [7] is more general than ours as he considered a more abstract setting. The main advantages of our method are:

- We prove that under some growth and coercivity conditions every critical point of the energy (1.5) is actually a minimizer and a solution of (1.6).
- Our result, giving the answer for question **(b)**, doesn't require any assumption of compactness or weak continuity of Λ_t (these assumptions are needed only for the proof of existence of minimizer, i.e., in connection with question **(a)**).
- Our method for answering question **(b)** uses only elementary arguments.

We can rewrite the definition of E_λ in (1.5) as follows. Since I is a self-adjoint and strictly positive operator, there exists a Hilbert space H and an injective bounded linear operator $T : X \to H$, whose image is dense in H, such that if we consider the linear operator $\widetilde{T} : H \to X^*$, defined by the formula

$$\big\langle x, \widetilde{T}\cdot y\big\rangle_{X\times X^*} := \big\langle T\cdot x, y\big\rangle_{H\times H} \qquad \text{for every } y \in H \text{ and } x \in X\,, \tag{1.7}$$

then we will have $\widetilde{T}\circ T\equiv I$, see Lemma 2.7 for details. We call $\{X,H,X^*\}$ an evolution triple with the corresponding inclusion operator $T:X\to H$ and $\widetilde{T}:H\to X^*$. Thus, if $v_0=\widetilde{T}\cdot w_0$, for some $w_0\in H$ and $p=q^*:=q/(q-1)$, where $q>1$, then we have

$$\int_0^{T_0}\left\langle u(t),\frac{d}{dt}\{I\cdot u(t)\}\right\rangle_{X\times X^*}dt=\frac{1}{2}\big\|T\cdot u(T_0)\big\|_H^2-\frac{1}{2}\big\|w_0\big\|_H^2$$

(see Lemma 2.8 for details) and therefore,

$$E_\lambda(u)=J(u):=$$
$$\int\limits_0^{T_0}\left\{\Psi_t\Big(\lambda u(t)\Big)+\Psi_t^*\Big(-\frac{d}{dt}\{I\cdot u(t)\}-\Lambda_t\big(u(t)\big)\Big)+\lambda\Big\langle u(t),\Lambda_t\big(u(t)\big)\Big\rangle_{X\times X^*}\right\}dt+\frac{\lambda}{2}\big\|T\cdot u(T_0)\big\|_H^2-\frac{\lambda}{2}\big\|w_0\big\|_H^2$$
$$\forall\, u(t)\in L^q\big((0,T_0);X\big)\ \text{ s.t. }\ I\cdot u(t)\in W^{1,q^*}\big((0,T_0);X^*\big)\ \text{ and }\ I\cdot u(0)=\widetilde{T}\cdot w_0 \tag{1.8}$$

Our first main result provides the answer for question **(b)**, under some coercivity and growth conditions on Ψ_t and Λ_t (see an equivalent formulation in Theorem 3.1 and Proposition 3.1):

Theorem 1.1. *Let $\{X,H,X^*\}$ be an evolution triple with the corresponding inclusion linear operators $T:X\to H$, which we assume to be injective and having dense image in H, $\widetilde{T}:H\to X^*$ be defined by (1.7) and $I:=\widetilde{T}\circ T:X\to X^*$. Next let $\lambda\in\{0,1\}$, $q\geq 2$, $p=q^*:=q/(q-1)$ and $w_0\in H$. Furthermore, for every $t\in[0,T_0]$ let $\Psi_t(x):X\to[0,+\infty)$ be a strictly convex function which is Gateaux differentiable at every $x\in X$, satisfying $\Psi_t(0)=0$ and the condition*

$$(1/C_0)\,\|x\|_X^q-C_0\leq\Psi_t(x)\leq C_0\,\|x\|_X^q+C_0\quad\forall x\in X,\ \forall t\in[0,T_0]\,, \tag{1.9}$$

for some $C_0>0$. We also assume that $\Psi_t(x)$ is a Borel function of its variables (x,t). Next, for every $t\in[0,T_0]$ let $\Lambda_t(x):X\to X^*$ be a function which is Gateaux differentiable at every $x\in X$, s.t. $\Lambda_t(0)\in L^{q^*}\big((0,T_0);X^*\big)$ and the derivative of Λ_t satisfies the growth condition

$$\|D\Lambda_t(x)\|_{\mathcal{L}(X;X^*)}\leq g\big(\|T\cdot x\|_H\big)\left(\|x\|_X^{q-2}+1\right)\quad\forall x\in X,\ \forall t\in[0,T_0]\,, \tag{1.10}$$

for some non-decreasing function $g(s):[0+\infty)\to(0,+\infty)$. We also assume that $\Lambda_t(x)$ is strongly Borel on the pair of variables (x,t) (see Definition 2.2). Assume also that Ψ_t and Λ_t satisfy the following monotonicity condition

$$\left\langle h,\lambda\Big\{D\Psi_t\big(\lambda x+h\big)-D\Psi_t(\lambda x)\Big\}+D\Lambda_t(x)\cdot h\right\rangle_{X\times X^*}\geq-\hat{g}\big(\|T\cdot x\|_H\big)\Big(\|x\|_X^q+\mu(t)\Big)\,\|T\cdot h\|_H^2$$
$$\forall x,h\in X,\ \forall t\in[0,T_0]\,, \tag{1.11}$$

for some non-decreasing function $\hat{g}(s):[0+\infty)\to(0,+\infty)$ and some nonnegative function $\mu(t)\in L^1\big((0,T_0);\mathbb{R}\big)$. Consider the set

$$\mathcal{R}_q:=\Big\{u(t)\in L^q\big((0,T_0);X\big):\ I\cdot u(t)\in W^{1,q^*}\big((0,T_0);X^*\big)\Big\}\,, \tag{1.12}$$

and the minimization problem

$$\inf\Big\{J(u):\ u(t)\in\mathcal{R}_q\ \ s.t\ \ I\cdot u(0)=\widetilde{T}\cdot w_0\Big\}\,, \tag{1.13}$$

where $J(u)$ is defined by (1.8). Then for every $u\in\mathcal{R}_q$ such that $I\cdot u(0)=\widetilde{T}\cdot w_0$ and for arbitrary function $h(t)\in\mathcal{R}_q$, such that $I\cdot h(0)=0$, the finite limit $\lim\limits_{s\to0}\big(J(u+sh)-J(u)\big)/s$ exists. Moreover, for every such u the following four statements are equivalent:

(1) *u is a critical point of (1.13), i.e., for any function $h(t)\in\mathcal{R}_q$, such that $I\cdot h(0)=0$ we have*

$$\lim_{s\to0}\frac{J(u+sh)-J(u)}{s}=0\,. \tag{1.14}$$

(2) *u is a minimizer to* (1.13).

(3) $J(u) = 0$.

(4) *u is a solution to*

$$\begin{cases} \frac{d}{dt}\big\{I \cdot u(t)\big\} + \Lambda_t\big(u(t)\big) + D\Psi_t\big(\lambda u(t)\big) = 0 & \text{in } (0, T_0), \\ I \cdot u(0) = \widetilde{T} \cdot w_0. \end{cases} \tag{1.15}$$

Finally there exists at most one function $u \in \mathcal{R}_q$ which satisfies (1.15).

Remark 1.1. Assume that, instead of (1.11), one requires that Ψ_t and Λ_t satisfy the following inequality

$$\Big\langle h, \lambda\Big\{D\Psi_t\big(\lambda x + h\big) - D\Psi_t(\lambda x)\Big\} + D\Lambda_t(x) \cdot h\Big\rangle_{X \times X^*} \geq \\ \frac{\|h\|_X^2}{\tilde{g}(\|T \cdot x\|_H)} - \tilde{g}\big(\|T \cdot x\|_H\big)\Big(\|x\|_X^q + \mu(t)\Big)^{(2-r)/2} \|h\|_X^r \, \|T \cdot h\|_H^{(2-r)} \quad \forall x, h \in X, \ \forall t \in [a, b], \tag{1.16}$$

for some non-decreasing function $\tilde{g}(s) : [0 + \infty) \to (0, +\infty)$, some nonnegative function $\mu(t) \in L^1\big((0, T_0); \mathbb{R}\big)$ and some constant $r \in (0, 2)$. Then (1.11) follows by the trivial inequality $(r/2)\, a^2 + \big((2-r)/2\big)\, b^2 \geq a^r\, b^{2-r}$.

Our first result about the existence of minimizer for $J(u)$ is the following Proposition (see Proposition 3.2 for an equivalent formulation):

Proposition 1.1. *Assume that $\{X, H, X^*\}$, $T, \widetilde{T}, I$, λ, q, p, Ψ_t and Λ_t satisfy all the conditions of Theorem 1.1 together with the assumption $\lambda = 1$. Moreover, assume that Ψ_t and Λ_t satisfy the following positivity condition*

$$\Psi_t(x) + \Big\langle x, \Lambda_t(x)\Big\rangle_{X \times X^*} \geq \frac{1}{\tilde{C}}\, \|x\|_X^q - \tilde{C}\Big(\|x\|_X^r + 1\Big)\Big(\|T \cdot x\|_H^{(2-r)} + 1\Big) - \bar{\mu}(t) \quad \forall x \in X, \ \forall t \in [0, T_0], \tag{1.17}$$

where $r \in [0, 2)$ and $\tilde{C} > 0$ are some constants and $\bar{\mu}(t) \in L^1\big((0, T_0); \mathbb{R}\big)$ is some nonnegative function. Furthermore, assume that

$$\Lambda_t(x) = A_t\big(S \cdot x\big) + \Theta_t(x) \qquad \forall\, x \in X, \ \forall\, t \in [0, T_0], \tag{1.18}$$

where Z is a Banach space, $S : X \to Z$ is a compact operator and for every $t \in [0, T_0]$ $A_t(z) : Z \to X^*$ is a function which is strongly Borel on the pair of variables (z, t) and Gateaux differentiable at every $z \in Z$, $\Theta_t(x) : X \to X^*$ is strongly Borel on the pair of variables (x, t) and Gateaux differentiable at every $x \in X$, $\Theta_t(0), A_t(0) \in L^{q^*}\big((0, T_0); X^*\big)$ and the derivatives of A_t and Θ_t satisfy the growth condition

$$\|D\Theta_t(x)\|_{\mathcal{L}(X;X^*)} + \|DA_t(S \cdot x)\|_{\mathcal{L}(Z;X^*)} \leq g\big(\|T \cdot x\|\big)\left(\|x\|_X^{q-2} + 1\right) \quad \forall x \in X, \ \forall t \in [0, T_0] \tag{1.19}$$

for some nondecreasing function $g(s) : [0, +\infty) \to (0 + \infty)$. Next assume that for every sequence $\big\{x_n(t)\big\}_{n=1}^{+\infty} \subset L^q\big((0, T_0); X\big)$ such that the sequence $\big\{I \cdot x_n(t)\big\}$ is bounded in $W^{1,q^*}\big((0, T_0); X^*\big)$ and $x_n(t) \rightharpoonup x(t)$ weakly in $L^q\big((0, T_0); X\big)$ we have

- *$\Theta_t\big(x_n(t)\big) \rightharpoonup \Theta_t\big(x(t)\big)$ weakly in $L^{q^*}\big((0, T_0); X^*\big)$,*
- *$\underline{\lim}_{n \to +\infty} \int_0^{T_0} \Big\langle x_n(t), \Theta_t\big(x_n(t)\big)\Big\rangle_{X \times X^*} dt \geq \int_0^{T_0} \Big\langle x(t), \Theta_t\big(x(t)\big)\Big\rangle_{X \times X^*} dt.$*

Finally let $w_0 \in H$ be such that $w_0 = T \cdot u_0$ for some $u_0 \in X$, or more generally, $w_0 \in H$ be such that $\mathcal{A}_{w_0} := \big\{u \in \mathcal{R}_q : I \cdot u(0) = \widetilde{T} \cdot w_0\big\} \neq \emptyset$. Then there exists a minimizer to (1.13).

As a consequence of Theorem 1.1 and Proposition 1.1 we have the following Corollary.

Corollary 1.1. *Assume that we are in the settings of Proposition 1.1. Then there exists a unique solution* $u(t) \in \mathcal{R}_q$ *to*

$$\begin{cases} \frac{d}{dt}\big\{I \cdot u(t)\big\} + \Lambda_t\big(u(t)\big) + D\Psi_t\big(u(t)\big) = 0 & \text{in } (0, T_0), \\ I \cdot u(0) = \widetilde{T} \cdot w_0. \end{cases} \tag{1.20}$$

In [15],using Corollary 1.1 as a basis, by the appropriate approximation, we obtain further existence Theorems, under much weaker assumption on coercivity and compactness. Moreover, applying these general Theorems, we provide with the existence results for various classes of time dependent partial differential equations including parabolic, hyperbolic, Shrödinger and Navier-Stokes systems.

In order to demonstrate the basic idea of the proof of the key Theorem 1.1 consider the simple example of the scalar parabolic equation of the following form:

$$\begin{cases} \partial_t u + \operatorname{div}_x F(u) - \Delta_x u = 0 & \forall x \in \Omega \subset\subset \mathbb{R}^N, \ \forall t \in (0, T_0) \\ u(x,t) = 0 \quad \forall x \in \partial\Omega, & \forall t \in (0, T_0) \\ u(x,0) = v_0(x) & \forall x \in \Omega, \end{cases} \tag{1.21}$$

where we assume $F : \mathbb{R} \to \mathbb{R}^N$ to be smooth and globally Lipschitz function. In this case we take $X := W_0^{1,2}(\Omega)$, $H := L^2(\Omega)$ and T to be a trivial embedding. Then $X^* = W^{-1,2}(\Omega)$ and the energy-functional takes the form

$$\bar{E}(u) := \frac{1}{2}\int_0^{T_0}\int_\Omega \left(|\nabla_x u|^2 + \left|\nabla_x\Big\{\Delta_x^{-1}\big(\partial_t u + \operatorname{div}_x F(u)\big)\Big\}\right|^2\right) dxdt + \frac{1}{2}\int_\Omega \Big(\big|u(x,T_0)\big|^2 - \big|u(x,0)\big|^2\Big) dx, \tag{1.22}$$

where $\Delta^{-1} f$ is the solution of

$$\begin{cases} \Delta y = f & x \in \Omega\,, \\ y = 0 & \forall x \in \partial\Omega\,. \end{cases}$$

Let us investigate the Euler-Lagrange equation for (1.22). If u satisfies $u(x,t) = 0$ for every $(x,t) \in \partial\Omega \times (0, T_0)$ and $u(x,0) = v_0(x)$, then,

$$\bar{E}(u) := \frac{1}{2}\int_0^{T_0}\int_\Omega \left(\left|\nabla_x\Big\{u - \Delta_x^{-1}\big(\partial_t u + \operatorname{div}_x F(u)\big)\Big\}\right|^2\right) dxdt. \tag{1.23}$$

Set $W_u := u - \Delta_x^{-1}\big(\partial_t u + \operatorname{div}_x F(u)\big)$. Thus, for every minimizer u of the energy (1.23) and for every smooth test function $\delta(x,t)$, satisfying $\delta(x,t) = 0$ for every $(x,t) \in \partial\Omega \times (0, T_0)$ and $\delta(x,0) \equiv 0$, we obtain

$$0 = \frac{d\bar{E}(u + s\delta)}{ds}\Big|_{(s=0)} = \lim_{s\to 0}\frac{1}{2s}\int_0^{T_0}\int_\Omega \Big(|\nabla_x W_{(u+s\delta)}|^2 - |\nabla_x W_u|^2\Big) =$$

$$-\lim_{s\to 0}\frac{1}{2s}\int_0^{T_0}\int_\Omega \big(\Delta_x W_{(u+s\delta)} - \Delta_x W_u\big)\cdot\big(W_{(u+s\delta)} + W_u\big) =$$

$$\lim_{s\to 0}\frac{1}{2}\int_0^{T_0}\int_\Omega \Big(-\Delta_x\delta + \partial_t\delta + \operatorname{div}_x\big(F(u+s\delta) - F(u)\big)/s\Big)\cdot\big(W_{(u+s\delta)} + W_u\big) =$$

$$\int_0^{T_0}\int_\Omega \Big(\nabla W_u \cdot \nabla_x\delta + W_u \cdot \partial_t\delta - \big(F'(u)\cdot\nabla_x W_u\big)\delta\Big)\,.$$

Since δ was arbitrary (in particular $\delta(x, T_0)$ is free) we deduce that $\Delta_x W_u + \partial_t W_u + F'(u)\cdot\nabla_x W_u = 0$, $W_u(x, T_0) = 0$ and $W_u = 0$ if $x \in \partial\Omega$. Changing variables $\tau := T_0 - t$ gives that W_u is a solution of the following linear parabolic equation with the trivial initial and boundary conditions:

$$\begin{cases} \partial_\tau W_u - F'(u)\cdot\nabla_x W_u = \Delta_x W_u & \forall (x,\tau) \in \Omega \times (0, T_0)\,, \\ W_u(x,0) = 0 & \forall x \in \Omega\,, \\ W_u(x,\tau) = 0 & \forall (x,\tau) \in \partial\Omega \times (0, T_0)\,. \end{cases}$$

Therefore $W_u = 0$ and then $\Delta_x u = \partial_t u + \operatorname{div}_x F(u)$, i.e., u is the solution of (1.21).

2 Notations and preliminaries

Throughout the paper by linear space we mean a real linear space.

- For given normed space X we denote by X^* the dual space (the space of continuous (bounded) linear functionals from X to $\mathbb{R}$).
- For given $h \in X$ and $x^* \in X^*$ we denote by $\langle h, x^* \rangle_{X \times X^*}$ the value in $\mathbb{R}$ of the functional x^* on the vector h.
- For given two normed linear spaces X and Y we denote by $\mathcal{L}(X;Y)$ the linear space of continuous (bounded) linear operators from X to Y.
- For given $A \in \mathcal{L}(X;Y)$ and $h \in X$ we denote by $A \cdot h \in Y$ the value of the operator A at the point h.
- We set $\|A\|_{\mathcal{L}(X;Y)} = \sup\{\|A \cdot h\|_Y : \ h \in X, \ \|h\|_X \leq 1\}$. Then it is well known that $\mathcal{L}(X;Y)$ will be a normed linear space. Moreover $\mathcal{L}(X;Y)$ will be a Banach space if Y is a Banach space.

Definition 2.1. Let X and Y be two normed linear spaces. We say that a function $F : X \to Y$ is Gateaux differentiable at the point $x \in X$ if there exists $A \in \mathcal{L}(X;Y)$ such that the following limit exists in Y and satisfy,

$$\lim_{s \to 0} \frac{1}{s}\Big(F(x+sh) - F(x)\Big) = A \cdot h \quad \forall h \in X\,.$$

In this case we denote the operator A by $DF(x)$ and the value $A \cdot h$ by $DF(x) \cdot h$.

Definition 2.2. Let X and Y be two normed linear spaces and $U \subset X$ be a Borel subset. We say that the mapping $F(x) : U \to Y$ is strongly Borel if the following two conditions are satisfied.

- F is a Borel mapping i.e. for every Borel set $W \subset Y$, the set $\{x \in U : \ F(x) \in W\}$ is also Borel.
- For every separable subspace $X' \subset X$, the set $\{y \in Y \ : \ y = F(x), \ x \in U \cap X'\}$ is also contained in some separable subspace of Y.

Definition 2.3. For a given Banach space X with the associated norm $\|\cdot\|_X$ and a real interval (a,b) we denote by $L^q(a,b;X)$ the linear space of (equivalence classes of) strongly measurable (i.e equivalent to some strongly Borel mapping) functions $f : (a,b) \to X$ such that the functional

$$\|f\|_{L^q(a,b;X)} := \begin{cases} \Big(\int_a^b \|f(t)\|_X^q dt\Big)^{1/q} & \text{if } \ 1 \leq q < \infty \\ \operatorname{es\,sup}_{t \in (a,b)} \|f(t)\|_X & \text{if } \ q = \infty \end{cases}$$

is finite. It is known that this functional defines a norm with respect to which $L^q(a,b;X)$ becomes a Banach space. Moreover, if X is reflexive and $1 < q < \infty$ then $L^q(a,b;X)$ will be a reflexive space with the corresponding dual space $L^{q^*}(a,b;X^*)$, where $q^* = q/(q-1)$. It is also well known that the subspace of continuous functions $C^0([a,b];X) \subset L^q(a,b;X)$ is dense i.e. for every $f(t) \in L^q(a,b;X)$ there exists a sequence $\{f_n(t)\} \subset C^0([a,b];X)$ such that $f_n(t) \to f(t)$ in the strong topology of $L^q(a,b;X)$.

We will need the following simple Lemma.

Lemma 2.1. *Let X be a Banach space, (a,b) be a bounded real interval and $f(t) \in L^q(a,b;X)$ for some $1 \leq q < +\infty$. Then if we denote*

$$\bar{f}(t) := \begin{cases} f(t) & \text{if } \ t \in (a,b)\,, \\ 0 & \text{if } \ t \notin (a,b)\,, \end{cases} \tag{2.1}$$

then

$$\lim_{h\to 0}\int_{\mathbb{R}}\left\|\bar{f}(t+h)-\bar{f}(t)\right\|_X^q dt=0\,, \tag{2.2}$$

and

$$\lim_{h\to 0}\int_{\mathbb{R}}\left(\frac{1}{h}\int_{-h}^{h}\left\|\bar{f}(t+\tau)-\bar{f}(t)\right\|_X^q d\tau\right)dt=0\,. \tag{2.3}$$

Moreover, for every sequence $\varepsilon_n\to 0^+$ as $n\to+\infty$, up to a subsequence, still denoted by ε_n we have

$$\lim_{n\to+\infty}\frac{1}{\varepsilon_n}\int_{-\varepsilon_n}^{\varepsilon_n}\left\|\bar{f}(t+\tau)-\bar{f}(t)\right\|_X^q d\tau=\lim_{n\to+\infty}\int_{-1}^{1}\left\|\bar{f}(t+\varepsilon_n s)-\bar{f}(t)\right\|_X^q ds=0\quad \textit{for a.e. } t\in\mathbb{R}\,. \tag{2.4}$$

See the proof of this Lemma in the end of the Appendix.

Definition 2.4. Let X be a reflexive Banach space and let (a,b) be a finite real interval. We say that $v(t)\in L^q(a,b;X)$ belongs to $W^{1,q}(a,b;X)$ if there exists $f(t)\in L^q(a,b;X)$ such that for every $\delta(t)\in C^1\big((a,b);X^*\big)$ satisfying $\operatorname{supp}\delta\subset\subset(a,b)$ we have

$$\int_a^b\big\langle f(t),\delta(t)\big\rangle_{X\times X^*}dt=-\int_a^b\Big\langle v(t),\frac{d\delta}{dt}(t)\Big\rangle_{X\times X^*}dt\,.$$

In this case we denote $f(t)$ by $v'(t)$ or by $\frac{dv}{dt}(t)$. It is well known that if $v(t)\in W^{1,1}(a,b;X)$ then $v(t)$ is a bounded and continuous function on $[a,b]$ (up to a redefining of $v(t)$ on a subset of $[a,b]$ of Lebesgue measure zero), i.e. $v(t)\in C^0\big([a,b];X\big)$ and for every $\delta(t)\in C^1\big([a,b];X^*\big)$ and every subinterval $[\alpha,\beta]\subset[a,b]$ we have

$$\int_\alpha^\beta\left\{\Big\langle\frac{dv}{dt}(t),\delta(t)\Big\rangle_{X\times X^*}+\Big\langle v(t),\frac{d\delta}{dt}(t)\Big\rangle_{X\times X^*}\right\}dt=\big\langle v(\beta),\delta(\beta)\big\rangle_{X\times X^*}-\big\langle v(\alpha),\delta(\alpha)\big\rangle_{X\times X^*}\,. \tag{2.5}$$

Lemma 2.2. *Let X and Y be two reflexive Banach spaces, $S\in\mathcal{L}(X,Y)$ be an injective operator (i.e. it satisfies $\ker S=0$) and (a,b) be a finite real interval. Then if $u(t)\in L^q(a,b;X)$ is such that $v(t):=S\cdot u(t)\in W^{1,q}(a,b;Y)$ and there exists $f(t)\in L^q(a,b;X)$ such that $\frac{dv}{dt}(t)=S\cdot f(t)$ then $u(t)\in W^{1,q}(a,b;X)$ and $\frac{du}{dt}(t)=f(t)$.*

Proof. By the definition for every $\delta(t)\in C^1\big((a,b);Y^*\big)$ satisfying $\operatorname{supp}\delta\subset\subset(a,b)$ we have

$$\int_a^b\Big\langle S\cdot f(t),\delta(t)\Big\rangle_{Y\times Y^*}dt=-\int_a^b\Big\langle S\cdot u(t),\frac{d\delta}{dt}(t)\Big\rangle_{Y\times Y^*}dt\,. \tag{2.6}$$

Next let $S^*\in\mathcal{L}(Y^*,X^*)$ be the adjoint to S operator defined by

$$\big\langle x,S^*\cdot y^*\big\rangle_{X\times X^*}:=\big\langle S\cdot x,y^*\big\rangle_{Y\times Y^*}\qquad\text{for every } y^*\in Y^* \text{ and } x\in X\,.$$

Then, by (2.6), for every $\delta(t)\in C^1\big((a,b);Y^*\big)$ satisfying $\operatorname{supp}\delta\subset\subset(a,b)$ we have $S^*\cdot\delta(t)\in C^1\big((a,b);X^*\big)$ and

$$\int_a^b\Big\langle f(t),S^*\cdot\delta(t)\Big\rangle_{X\times X^*}dt=-\int_a^b\Big\langle u(t),S^*\cdot\frac{d\delta}{dt}(t)\Big\rangle_{X\times X^*}dt=-\int_a^b\Big\langle u(t),\frac{d\big(S^*\cdot\delta(t)\big)}{dt}\Big\rangle_{X\times X^*}dt\,. \tag{2.7}$$

However, S is an injective operator and thus S^* must have dense range in X^*. Therefore for every $\varphi(t)\in C^1\big((a,b);X^*\big)$ satisfying $\operatorname{supp}\varphi\subset\subset(a,b)$ there exists a sequence $\{\delta_n(t)\}\subset C^1\big((a,b);Y^*\big)$

satisfying $\operatorname{supp}\delta_n \subset\subset (a,b)$ and such that $S^*\cdot\delta_n(t)\to\varphi(t)$ in X^* uniformly by t and $S^*\cdot\frac{d\delta_n}{dt}(t)\to\frac{d\varphi}{dt}(t)$ in X^* uniformly by t. Therefore, by (2.7) we deduce

$$\int_a^b \Big\langle f(t),\varphi(t)\Big\rangle_{X\times X^*}dt=-\int_a^b \Big\langle u(t),\frac{d\varphi}{dt}(t)\Big\rangle_{X\times X^*}dt\,,$$

for every $\varphi(t)\in C^1\big((a,b);X^*\big)$ satisfying $\operatorname{supp}\varphi\subset\subset(a,b)$. This completes the proof. $\square$

Definition 2.5. Let X be a Banach space. We say that a function $\Psi(x):X\to\mathbb{R}$ is convex (strictly convex) if for every $\lambda\in(0,1)$ and for every $x,y\in X$ s.t. $x\neq y$ we have

$$\Psi\Big(\lambda x+(1-\lambda)y\Big)\ \le\ (\ <\ )\quad \lambda\Psi(x)+(1-\lambda)\Psi(y)\,.$$

It is well known that if $\Psi(x):X\to\mathbb{R}$ is a convex (strictly convex) function which is Gateaux differentiable at every $x\in X$ then for every $x,y\in X$ s.t. $x\neq y$ we have

$$\Psi(y)\ \ \ge\ (\ >\ )\quad \Psi(x)+\Big\langle y-x,D\Psi(x)\Big\rangle_{X\times X^*}\,, \tag{2.8}$$

and

$$\Big\langle y-x,D\Psi(y)-D\Psi(x)\Big\rangle_{X\times X^*}\ \ \ge\ (\ >\ )\quad 0\,, \tag{2.9}$$

(remember that $D\Psi(x)\in X^*$). Furthermore, Ψ is weakly lower semicontinuous on X. Moreover, if some function $\Psi(x):X\to\mathbb{R}$ is Gateaux differentiable at every $x\in X$ and satisfy either (2.8) or (2.9) for every $x,y\in X$ s.t. $x\neq y$, then $\Psi(y)$ is convex (strictly convex).

Definition 2.6. Let X be a reflexive Banach space and let $\Psi(x):X\to\mathbb{R}$ be a convex function. For every $y\in X^*$ set the Legendre transform of Ψ by

$$\Psi^*(y):=\sup\Big\{\big\langle z,y\big\rangle_{X\times X^*}-\Psi(z):\ z\in X\Big\}\,.$$

Lemma 2.3. *Let X be a reflexive Banach space and let $\Psi(x):X\to[0,+\infty)$ be a strictly convex function which is Gateaux differentiable at every $x\in X$ and satisfies $\Psi(0)=0$ and*

$$\lim_{\|x\|_X\to+\infty}\frac{1}{\|x\|_X}\Psi(x)=+\infty\,. \tag{2.10}$$

Then $\Psi^*(y)$ is a strictly convex function from X^* to $[0,+\infty)$ and satisfies $\Psi^*(0)=0$. Furthermore, $\Psi^*(y)$ is Gateaux differentiable at every $y\in X^*$. Moreover $x\in X$ satisfies $x=D\Psi^*(y)$ (remember that $D\Psi^*(y)\in X^{**}=X$) if and only if $y\in X^*$ satisfies $y=D\Psi(x)$ (remember that $D\Psi(x)\in X^*$). Finally if in addition Ψ satisfies

$$(1/C_0)\,\|x\|_X^q-C_0\le\Psi(x)\le C_0\,\|x\|_X^q+C_0\quad\forall x\in X\,, \tag{2.11}$$

for some $q>1$ and $C_0>0$, then

$$(1/C)\,\|y\|_{X^*}^{q^*}-C\le\Psi^*(y)\le C\,\|y\|_{X^*}^{q^*}+C\quad\forall y\in X^*\,, \tag{2.12}$$

for some $C>0$ depending only on C_0 and q, where $q^*:=q/(q-1)$. Moreover, for some $\bar C_0,\bar C>0$, that depend only on C and q from (2.11), we have

$$\|D\Psi(x)\|_{X^*}\le\bar C_0\|x\|_X^{q-1}+\bar C_0\quad\forall x\in X\,, \tag{2.13}$$

and

$$\|D\Psi^*(y)\|_X\le\bar C\|y\|_{X^*}^{q^*-1}+\bar C\quad\forall y\in X^*\,. \tag{2.14}$$

Proof. First since $\Psi(0)=0$ it is clear that for every $y\in X^*$ we have $\Psi^*(y)\geq 0$. Next since for every $x\in X$ we have $\Psi(x)\geq 0$ then $\Psi^*(0)\leq 0$ and so $\Psi^*(0)=0$. Next by the growth condition (2.10) we deduce that $\Psi^*(y)<+\infty$ for every $y\in X^*$. So $\Psi^*(y):X^*\to[0,+\infty)$. Moreover, it easy follows from the definition of Legendre transform, that $\Psi^*(y)$ is a convex function on X^*.

Next since Ψ is weakly lower semicontinuous on X and satisfies growth condition (2.10) then for every $y\in X^*$ there exists $z_y\in X$ such that

$$\Psi(z_y)-\left\langle z_y,y\right\rangle_{X\times X^*}=\inf\left\{\Psi(z)-<z,y>_{X\times X^*}:\ z\in X\right\},\tag{2.15}$$

i.e.

$$\Psi^*(y):=\left\langle z_y,y\right\rangle_{X\times X^*}-\Psi(z_y)\,.\tag{2.16}$$

Moreover we have

$$D\Psi(z_y)=y\,,\tag{2.17}$$

However, since Ψ is a strictly convex function, by (2.9) for every $z\in X$ s.t. $z\neq z_y$ we must have

$$\left\langle z-z_y,D\Psi(z)-D\Psi(z_y)\right\rangle_{X\times X^*}>0\,.\tag{2.18}$$

Therefore, in particular for every $y\in X^*$ $z=z_y$ is a unique solution of the equation $D\Psi(z)=y$ and

$$\left\langle z_{y_1}-z_{y_2},y_1-y_2\right\rangle_{X\times X^*}>0\,,\tag{2.19}$$

for every $y_1,y_2\in X^*$ s.t. $y_1\neq y_2$. Next let $y_0,h\in X^*$. Then by the definition of Ψ^* for every $s\in\mathbb{R}$ we have

$$s\left\langle z_{y_0},h\right\rangle_{X\times X^*}\leq\Psi^*(y_0+sh)-\Psi^*(y_0)\leq s\left\langle z_{(y_0+sh)},h\right\rangle_{X\times X^*}.\tag{2.20}$$

On the other hand by (2.15) we have $\Psi\big(z_{(y_0+sh)}\big)\leq\left\langle z_{y_0+sh},y_0+sh\right\rangle_{X\times X^*}$. Therefore, using growth condition (2.10) we deduce that there exists $\bar C>0$ such that $\|z_{y_0+sh}\|_X\leq\bar C$ for every $s\in(-1,1)$. Thus using the fact that X is reflexive we deduce that for any sequence $\{s_n\}_{n=1}^{+\infty}\subset(-1,1)$ such that $\lim_{n\to+\infty}s_n=0$, up to a subsequence, we must have $z_{y_0+s_nh}\rightharpoonup\tilde z$ weakly in X. However, by (2.15) we have

$$\Psi\big(z_{(y_0+s_nh)}\big)-\left\langle z_{(y_0+s_nh)},y_0+s_nh\right\rangle_{X\times X^*}\leq\Psi(z)-<z,y_0+s_nh>_{X\times X^*}\quad\forall\,z\in X\,.\tag{2.21}$$

Then tending $n\to+\infty$ in (2.21), using the fact that, up to a subsequence, $z_{y_0+s_nh}\rightharpoonup\tilde z$ and that Ψ is weakly lower semicontinuous function we deduce that

$$\Psi(\tilde z)-<\tilde z,y_0>_{X\times X^*}\leq\Psi(z)-<z,y_0>_{X\times X^*}\quad\forall\,z\in X\,.\tag{2.22}$$

So $\tilde z$ is a minimizer to (2.15) with $y=y_0$ and therefore, $D\Psi(\tilde z)=y_0$. On the other hand $z=z_{y_0}$ is a unique solution of the equation $D\Psi(z)=y_0$. Therefore, $\tilde z=z_{y_0}$. So by (2.20), up to a subsequence, we have

$$\frac{1}{s_n}\Big(\Psi^*(y_0+s_nh)-\Psi^*(y_0)\Big)\to\left\langle z_{y_0},h\right\rangle_{X\times X^*}.\tag{2.23}$$

Since the sequence s_n was chosen arbitrary we deduce that

$$\lim_{s\to0}\frac{1}{s}\Big(\Psi^*(y_0+sh)-\Psi^*(y_0)\Big)\to\left\langle z_{y_0},h\right\rangle_{X\times X^*}.\tag{2.24}$$

Finally $y_0,h\in X$ also were chosen arbitrary and therefore we deduce that $\Psi^*(y)$ is Gateaux differentiable at every $y\in X^*$ and $D\Psi^*(y)=z_y$. Thus since $z=z_y$ is a unique solution of the equation $D\Psi(z)=y$ we deduce that $D\Psi^*(y)=z$ if and only if $D\Psi(z)=y$. Moreover by (2.19) we deduce that

$$\left\langle D\Psi^*(y_1)-D\Psi^*(y_2),y_1-y_2\right\rangle_{X\times X^*}>0\,,\tag{2.25}$$

for every $y_1,y_2\in X^*$ such that $y_1\neq y_2$. So Ψ^* is a strictly convex on X^* function.

Next if we consider function $\zeta(y):X^*\to\mathbb{R}$ defined by

$$\zeta(y):=\sup\left\{<z,y>_{X\times X^*}-k\|z\|_X^q:\ z\in X\right\},$$

for some $k > 0$, then

$$\zeta(y) = \sup\Big\{t < z, y >_{X\times X^*} - k|t|^q :\ t \in \mathbb{R},\ z \in X,\ \|z\|_X = 1\Big\} =$$
$$\sup\Big\{K| < z, y >_{X\times X^*} |^{q^*} :\ z \in X,\ \|z\|_X = 1\Big\} = K\|y\|^{q^*}_{X^*}\,,$$

for some $K > 0$ depending only on k and q. Thus using growth condition (2.11) and the definition of Ψ^* we easily deduce growth condition (2.12). So it remains to prove that growth condition (2.13) follows from growth condition (2.11) and (2.14) follows from (2.12). Indeed since Ψ is convex, from (2.8), for every $x, h \in X$ we have

$$\big\langle h, D\Psi(x)\big\rangle_{X\times X^*} \leq \Psi(x+h) - \Psi(x)\,. \tag{2.26}$$

Therefore, for every $x, h \in X$ such that $\|h\|_X \leq 1$ and $\|x\|_X \geq 1$ we have

$$\big\langle h, D\Psi(x)\big\rangle_{X\times X^*} \leq \frac{1}{\|x\|_X}\Big(\Psi\big(x + \|x\|_X h\big) - \Psi(x)\Big)\,. \tag{2.27}$$

Thus using growth condition (2.11) we deduce that for every $x, h \in X$ such that $\|h\|_X \leq 1$ and $\|x\|_X \geq 1$ we have

$$\big\langle h, D\Psi(x)\big\rangle_{X\times X^*} \leq \tilde{C}\|x\|^{q-1}_X\,, \tag{2.28}$$

and so

$$\|D\Psi(x)\|_{X^*} \leq \tilde{C}\|x\|^{q-1}_X\,, \tag{2.29}$$

for every x which satisfy $\|x\|_X \geq 1$. However, by (2.26) and (2.11) we have

$$\big\langle h, D\Psi(x)\big\rangle_{X\times X^*} \leq \hat{C}\,, \tag{2.30}$$

for every $x, h \in X$ such that $\|x\|_X \leq 1$ and $\|h\|_X \leq 1$, where $\hat{C} > 0$ is a constant. So $\|D\Psi(x)\|_{X^*} \leq \hat{C}$ for every x which satisfy $\|x\|_X \leq 1$. This together with (2.29) gives the desired result (2.13). Finally, Ψ^* is a convex on X^* and satisfy (2.12). Therefore, (2.14) follows exactly by the same way. □

Definition 2.7. Let Z be a Banach space and Z^* be a corresponding dual space. We say that the mapping $\Lambda(z) : Z \to Z^*$ is monotone (strictly monotone) if we have

$$\Big\langle y - z, \Lambda(y) - \Lambda(z)\Big\rangle_{Z\times Z^*} \geq (>)\, 0 \quad \forall\, y \neq z \in Z\,. \tag{2.31}$$

Definition 2.8. Let Z be a Banach space and Z^* be a corresponding dual space. We say that the mapping $\Lambda(z) : Z \to Z^*$ is pseudo-monotone if for every sequence $\{z_n\}_{n=1}^{+\infty} \subset Z$, satisfying

$$z_n \rightharpoonup z \ \text{ weakly in } Z \qquad \text{and} \qquad \overline{\lim_{n\to+\infty}} \Big\langle z_n - z, \Lambda(z_n)\Big\rangle_{Z\times Z^*} \leq 0 \tag{2.32}$$

we have

$$\underline{\lim}_{n\to+\infty} \Big\langle z_n - y, \Lambda(z_n)\Big\rangle_{Z\times Z^*} \geq \Big\langle z - y, \Lambda(z)\Big\rangle_{Z\times Z^*} \quad \forall y \in Z\,. \tag{2.33}$$

Lemma 2.4. *Let Z be a Banach space and Z^* be a corresponding dual space. Then the mapping $\Lambda(z) : Z \to Z^*$ is pseudo-monotone if and only if it satisfies the following conditions:*

(i) *For every sequence $\{z_n\}_{n=1}^{+\infty} \subset Z$, such that $z_n \rightharpoonup z$ weakly in Z we have*

$$\underline{\lim}_{n\to+\infty} \Big\langle z_n - z, \Lambda(z_n)\Big\rangle_{Z\times Z^*} \geq 0\,. \tag{2.34}$$

(ii) *If for some sequence $\{z_n\}_{n=1}^{+\infty} \subset Z$, such that $z_n \rightharpoonup z$ weakly in Z we have*

$$\lim_{n\to+\infty} \Big\langle z_n - z, \Lambda(z_n)\Big\rangle_{Z\times Z^*} = 0\,, \tag{2.35}$$

then $\Lambda(z_n) \rightharpoonup \Lambda(z)$ weakly* in Z^*.

Proof. Assume that the mapping $\Lambda(z) : Z \to Z^*$ is pseudo-monotone. Choose arbitrary sequence $z_n \rightharpoonup z$ weakly in Z and denote

$$K = \varliminf_{n\to+\infty} \Big\langle z_n - z, \Lambda(z_n)\Big\rangle_{Z\times Z^*}.$$

Then, up to a subsequence, still denoted by z_n we have

$$K = \lim_{n\to+\infty} \Big\langle z_n - z, \Lambda(z_n)\Big\rangle_{Z\times Z^*}.$$

Thus if we assume that $K \le 0$, by (2.33) with $y = z$ we deduce $K \ge 0$. Therefore, since the sequence $z_n \rightharpoonup z$ was chosen arbitrary we deduce that for every sequence $\{z_n\}_{n=1}^{+\infty} \subset Z$, such that $z_n \rightharpoonup z$ weakly in Z we have (2.34). Next assume that for some sequence $\{z_n\}_{n=1}^{+\infty} \subset Z$, such that $z_n \rightharpoonup z$ weakly in Z we have (2.35). Then, by (2.33) for this sequence we must have

$$\lim_{n\to+\infty} \Big\langle z_n - z, \Lambda(z_n)\Big\rangle_{Z\times Z^*} = 0 \quad \text{and} \quad \varliminf_{n\to+\infty} \Big\langle z_n - y, \Lambda(z_n)\Big\rangle_{X\times X^*} \ge \Big\langle z - y, \Lambda(z)\Big\rangle_{Z\times Z^*} \quad \forall y \in Z. \tag{2.36}$$

Therefore, plugging the first equality in (2.36) into the second inequality we obtain

$$\varliminf_{n\to+\infty} \Big\langle z - y, \Lambda(z_n)\Big\rangle_{Z\times Z^*} \ge \Big\langle z - y, \Lambda(z)\Big\rangle_{Z\times Z^*} \quad \forall y \in Z. \tag{2.37}$$

We can rewrite (2.37) as

$$\varliminf_{n\to+\infty} \big\langle h, \Lambda(z_n)\big\rangle_{Z\times Z^*} \ge \big\langle h, \Lambda(z)\big\rangle_{Z\times Z^*} \quad \forall h \in Z. \tag{2.38}$$

Thus, interchanging between h and $-h$ in (2.38) we obtain

$$\varlimsup_{n\to+\infty} \big\langle h, \Lambda(z_n)\big\rangle_{Z\times Z^*} \le \big\langle h, \Lambda(z)\big\rangle_{Z\times Z^*} \quad \forall h \in Z. \tag{2.39}$$

So, by plugging (2.38) and (2.39) we finally deduce

$$\lim_{n\to+\infty} \big\langle h, \Lambda(z_n)\big\rangle_{Z\times Z^*} = \big\langle h, \Lambda(z)\big\rangle_{Z\times Z^*} \quad \forall h \in Z. \tag{2.40}$$

I.e. $\Lambda(z_n) \rightharpoonup \Lambda(z)$ weakly* in Z^*.

Next assume that the mapping $\Lambda(z) : Z \to Z^*$ satisfies the conditions **(i)** and **(ii)**. Consider the sequence $\{z_n\}_{n=1}^{+\infty} \subset Z$, satisfying

$$z_n \rightharpoonup z \ \text{ weakly in } \ Z \qquad \text{and} \qquad \varlimsup_{n\to+\infty} \Big\langle z_n - z, \Lambda(z_n)\Big\rangle_{Z\times Z^*} \le 0$$

Then by condition **(i)** we must have

$$\lim_{n\to+\infty} \Big\langle z_n - z, \Lambda(z_n)\Big\rangle_{Z\times Z^*} = 0. \tag{2.41}$$

Thus by condition **(ii)** we must have

$$\lim_{n\to+\infty} \Big\langle z - y, \Lambda(z_n)\Big\rangle_{Z\times Z^*} = \Big\langle z - y, \Lambda(z)\Big\rangle_{Z\times Z^*} \quad \forall y \in Z. \tag{2.42}$$

Thus by (2.41) and (2.42) we finally deduce

$$\begin{aligned}\lim_{n\to+\infty} \Big\langle z_n - y, \Lambda(z_n)\Big\rangle_{Z\times Z^*} &= \lim_{n\to+\infty} \Big\langle z_n - z, \Lambda(z_n)\Big\rangle_{Z\times Z^*} + \lim_{n\to+\infty} \Big\langle z - y, \Lambda(z_n)\Big\rangle_{Z\times Z^*} \\ &= 0 + \Big\langle z - y, \Lambda(z)\Big\rangle_{Z\times Z^*} \qquad \forall y \in Z.\end{aligned} \tag{2.43}$$

Thus, the mapping $\Lambda(z) : Z \to Z^*$ is pseudo-monotone. □

Lemma 2.5. *Let Z be a Banach space and Z^* be a corresponding dual space. Assume that the mapping $\Lambda(z) : Z \to Z^*$ is monotone. Moreover assume that $\Lambda(z) : Z \to Z^*$ is continuous for every $z \in Z$ or more generally the function $\zeta_{z,h}(t) : \mathbb{R} \to \mathbb{R}$, defined by*

$$\zeta_{z,h}(t) := \Big\langle h, \Lambda\big(z - th\big)\Big\rangle_{Z\times Z^*} \quad \forall z, h \in Z\,, \quad \forall t \in \mathbb{R}\,, \tag{2.44}$$

is continuous on t for every $z, h \in Z$. Then the mapping $\Lambda(z)$ is pseudo-monotone.

Proof. Assume that the mapping $\Lambda(z) : Z \to Z^*$ is monotone. I.e.

$$\Big\langle y - z, \Lambda(y) - \Lambda(z)\Big\rangle_{Z\times Z^*} \geq 0 \quad \forall\, y, z \in Z\,. \tag{2.45}$$

Then in particular for every sequence $\{z_n\}_{n=1}^{+\infty} \subset Z$, such that $z_n \rightharpoonup z$ weakly in Z, we obtain

$$\varliminf_{n\to+\infty} \Big\langle z_n - z, \Lambda(z_n)\Big\rangle_{Z\times Z^*} \geq \varliminf_{n\to+\infty} \Big\langle z_n - z, \Lambda(z)\Big\rangle_{Z\times Z^*} = 0\,. \tag{2.46}$$

So the condition **(i)** of Lemma 2.4 is satisfied. Next assume that the sequence $\{z_n\}_{n=1}^{+\infty} \subset Z$ satisfies

$$z_n \rightharpoonup z \ \text{ weakly in } \ Z \quad \text{and} \quad \lim_{n\to+\infty} \Big\langle z_n - z, \Lambda(z_n)\Big\rangle_{Z\times Z^*} = 0\,. \tag{2.47}$$

We will prove now that we must have $\Lambda(z_n) \rightharpoonup \Lambda(z)$ weakly* in Z^*. Indeed, by (2.45) we obtain

$$\varliminf_{n\to+\infty} \Big\langle z_n - y, \Lambda(z_n) - \Lambda(y)\Big\rangle_{Z\times Z^*} \geq 0 \quad \forall\, y \in Z\,. \tag{2.48}$$

Thus plugging (2.47) into (2.48) we deduce

$$\varliminf_{n\to+\infty} \Big\langle z - y, \Lambda(z_n)\Big\rangle_{Z\times Z^*} = \varliminf_{n\to+\infty} \Big\langle z_n - y, \Lambda(z_n)\Big\rangle_{Z\times Z^*} \geq \\ \varliminf_{n\to+\infty} \Big\langle z_n - y, \Lambda(y)\Big\rangle_{Z\times Z^*} = \Big\langle z - y, \Lambda(y)\Big\rangle_{Z\times Z^*} \quad \forall\, y \in Z\,. \tag{2.49}$$

Then choosing $y := z - th$ in (2.49) for arbitrary $h \in Z$ and $t > 0$ we obtain

$$\varliminf_{n\to+\infty} \Big\langle h, \Lambda(z_n)\Big\rangle_{Z\times Z^*} \geq \Big\langle h, \Lambda(z - th)\Big\rangle_{Z\times Z^*} \quad \forall\, h \in Z\,,\ \forall t > 0\,. \tag{2.50}$$

Therefore, tending $t \to 0^+$ in (2.50) and using the continuity of the function in the r.h.s. of (2.50) we infer

$$\varliminf_{n\to+\infty} \big\langle h, \Lambda(z_n)\big\rangle_{Z\times Z^*} \geq \big\langle h, \Lambda(z)\big\rangle_{Z\times Z^*} \quad \forall\, h \in Z\,. \tag{2.51}$$

Thus, as before, interchanging between h and $-h$ in (2.51) we obtain

$$\varlimsup_{n\to+\infty} \big\langle h, \Lambda(z_n)\big\rangle_{Z\times Z^*} \leq \big\langle h, \Lambda(z)\big\rangle_{Z\times Z^*} \quad \forall h \in Z\,. \tag{2.52}$$

So, by plugging (2.51) and (2.52) we finally deduce

$$\lim_{n\to+\infty} \big\langle h, \Lambda(z_n)\big\rangle_{Z\times Z^*} = \big\langle h, \Lambda(z)\big\rangle_{Z\times Z^*} \quad \forall h \in Z\,. \tag{2.53}$$

I.e. $\Lambda(z_n) \rightharpoonup \Lambda(z)$ weakly* in Z^*. So the condition **(ii)** of Lemma 2.4 is satisfied. Therefore, by this Lemma the mapping $\Lambda(z)$ is pseudo-monotone. □

Lemma 2.6. *Let Y and Z be two reflexive Banach spaces. Furthermore, let $S \in \mathcal{L}(Y; Z)$ be an injective operator (i.e. it satisfies* $\ker S = \{0\}$*) and let $S^* \in \mathcal{L}(Z^*; Y^*)$ be the corresponding adjoint operator, which satisfies*

$$\big\langle y, S^* \cdot z^*\big\rangle_{Y\times Y^*} := \big\langle S \cdot y, z^*\big\rangle_{Z\times Z^*} \qquad \text{for every } z^* \in Z^* \text{ and } y \in Y\,. \tag{2.54}$$

Next assume that $a,b\in\mathbb{R}$ s.t. $a<b$. Let $w(t)\in L^\infty(a,b;Y)$ be such that the function $v:[a,b]\to Z$ defined by $v(t):=S\cdot\big(w(t)\big)$ belongs to $W^{1,q}(a,b;Z)$ for some $q\geq 1$. Then we can redefine w on a subset of $[a,b]$ of Lebesgue measure zero, so that $w(t)$ will be Y-weakly continuous in t on $[a,b]$ (i.e. $w\in C^0_w(a,b;Y)$). Moreover, for every $a\leq\alpha<\beta\leq b$ and for every $\delta(t)\in C^1\big([a,b];Z^*\big)$ we will have

$$\int\limits_\alpha^\beta\bigg\{\Big\langle\frac{dv}{dt}(t),\delta(t)\Big\rangle_{Z\times Z^*}+\Big\langle v(t),\frac{d\delta}{dt}(t)\Big\rangle_{Z\times Z^*}\bigg\}dt=\big\langle w(\beta),S^*\cdot\delta(\beta)\big\rangle_{Y\times Y^*}-\big\langle w(\alpha),S^*\cdot\delta(\alpha)\big\rangle_{Y\times Y^*}\,. \tag{2.55}$$

Proof. It is well known from the functional analysis that if Y and Z are reflexive Banach spaces and the operator $S\in\mathcal{L}(Y;Z)$ is injective then the corresponding adjoint operator $S^*\in\mathcal{L}(Z^*;Y^*)$ has dense image in Y^*. Next as it already was mentioned as a comment to Definition 2.4 that since $v(t)\in W^{1,q}(a,b;Z)$ we have $v(t)\in C^0\big([a,b];Z\big)$ (up to a redefining of $v(t)$ on a subset of $[a,b]$ of Lebesgue measure zero) and for every $\delta(t)\in C^1\big([a,b];Z^*\big)$ and every subinterval $[\alpha,\beta]\subset[a,b]$ we have

$$\int\limits_\alpha^\beta\bigg\{\Big\langle\frac{dv}{dt}(t),\delta(t)\Big\rangle_{Z\times Z^*}+\Big\langle v(t),\frac{d\delta}{dt}(t)\Big\rangle_{Z\times Z^*}\bigg\}dt=\big\langle v(\beta),\delta(\beta)\big\rangle_{Z\times Z^*}-\big\langle v(\alpha),\delta(\alpha)\big\rangle_{Z\times Z^*}\,. \tag{2.56}$$

Moreover, there exists set $\mathcal{A}\subset[a,b]$ such that $\mathcal{L}^1\big([a,b]\setminus\mathcal{A}\big)=0$ and $v(t):=S\cdot\big(w(t)\big)$ for every $t\in\mathcal{A}$. On the other hand, for every $l\in[a,b]$ we have $v(t)\rightharpoonup v(l)$ weakly in Z as $t\in[a,b]\to l^\pm$. Thus for every $l\in[a,b]$ and for every sequence $\{t_n\}_{n=1}^{+\infty}\subset\mathcal{A}$ such that $\lim_{n\to+\infty}t_n=l$ we have

$$\begin{aligned}\lim_{n\to+\infty}\big\langle w(t_n),S^*\cdot h\big\rangle_{Y\times Y^*}&=\lim_{n\to+\infty}\big\langle S\cdot w(t_n),h\big\rangle_{Z\times Z^*}\\&=\lim_{n\to+\infty}\big\langle v(t_n),h\big\rangle_{Z\times Z^*}=\big\langle v(l),h\big\rangle_{Z\times Z^*}\qquad\forall h\in Z^*\,.\end{aligned} \tag{2.57}$$

However, since $w(t)\in L^\infty(a,b;Y)$ and since Y is a reflexive space, we obtain that there exists $\bar w(l)\in Y$ such that, up to a subsequence of $\{t_n\}$, still denoted by $\{t_n\}$, we have $w(t_n)\rightharpoonup\bar w(l)$ weakly in Y. Plugging it into (2.57) we deduce

$$\big\langle S\cdot\bar w(l),h\big\rangle_{Z\times Z^*}=\big\langle\bar w(l),S^*\cdot h\big\rangle_{Y\times Y^*}=\big\langle v(l),h\big\rangle_{Z\times Z^*}\qquad\forall h\in Z^*\,. \tag{2.58}$$

I.e.

$$S\cdot\bar w(t)=v(t)\qquad\forall t\in[a,b]\,. \tag{2.59}$$

Moreover, clearly $\bar w(t)\in L^\infty(a,b;Y)$. On the other hand since $v(t):=S\cdot\big(w(t)\big)$ for every $t\in\mathcal{A}$ and since S is injective we obtain

$$\bar w(t)=w(t)\qquad\text{for a.e. }\;t\in[a,b]\,. \tag{2.60}$$

However, since for every $l\in[a,b]$ we have $v(t)\rightharpoonup v(l)$ weakly in Z as $t\in[a,b]\to l^\pm$, we infer

$$\begin{aligned}\lim_{t\to l^\pm}\big\langle\bar w(t),S^*\cdot h\big\rangle_{Y\times Y^*}=\lim_{t\to l^\pm}\big\langle S\cdot\bar w(t),h\big\rangle_{Z\times Z^*}=\lim_{t\to l^\pm}\big\langle v(t),h\big\rangle_{Z\times Z^*}=\\\big\langle v(l),h\big\rangle_{Z\times Z^*}=\big\langle S\cdot\bar w(l),h\big\rangle_{Z\times Z^*}=\big\langle\bar w(l),S^*\cdot h\big\rangle_{Y\times Y^*}\qquad\forall h\in Z^*\,.\end{aligned} \tag{2.61}$$

Therefore, since S^* has dense image in Y^* and since $\bar w(t)\in L^\infty(a,b;Y)$ we deduce $\bar w(t)\rightharpoonup\bar w(l)$ weakly in Y. Finally plugging (2.59) into (2.56) we obtain (2.55) with $\bar w$. □

Definition 2.9. Let X be a reflexive Banach space and X^* the corresponding dual space. Furthermore let H be a Hilbert space and $T\in\mathcal{L}(X,H)$ be an injective (i.e. it satisfies $\ker T=\{0\}$) inclusion operator such that its image is dense on H. Then we call the triple $\{X,H,X^*\}$ an evolution triple with the corresponding inclusion operator T. Throughout this paper we assume the space H^* be equal to H (remember that H is a Hilbert space) but in general we don't associate X^* with X even

in the case where X is a Hilbert space (and thus X^* will be isomorphic to X). Further we define the bounded linear operator $\widetilde{T} \in \mathcal{L}(H; X^*)$ by the formula

$$\big\langle x, \widetilde{T} \cdot y \big\rangle_{X \times X^*} := \big\langle T \cdot x, y \big\rangle_{H \times H} \qquad \text{for every } y \in H \text{ and } x \in X\,. \tag{2.62}$$

In particular $\|\widetilde{T}\|_{\mathcal{L}(H;X^*)} = \|T\|_{\mathcal{L}(X;H)}$ and since we assumed that the image of T is dense in H we deduce that $\ker \widetilde{T} = \{0\}$ and so $\widetilde{T}$ is an injective operator. So $\widetilde{T}$ is an inclusion of H to X^* and the operator $I := \widetilde{T} \circ T$ is an injective inclusion of X to X^*. Furthermore, clearly

$$\big\langle x, I \cdot z \big\rangle_{X \times X^*} = \big\langle T \cdot x, T \cdot z \big\rangle_{H \times H} = \big\langle z, I \cdot x \big\rangle_{X \times X^*} \qquad \text{for every } x, z \in X\,. \tag{2.63}$$

So $I \in \mathcal{L}(X, X^*)$ is self-adjoint operator. Moreover, I is strictly positive, since

$$\big\langle x, I \cdot x \big\rangle_{X \times X^*} = \|T \cdot x\|_H^2 > 0 \qquad \forall x \neq 0 \in X\,. \tag{2.64}$$

Lemma 2.7. *Let X be a reflexive Banach space and X^* the corresponding dual space. Furthermore let $I \in \mathcal{L}(X, X^*)$ be a self-adjoint and strictly positive operator. i.e.*

$$\big\langle x, I \cdot z \big\rangle_{X \times X^*} = \big\langle z, I \cdot x \big\rangle_{X \times X^*} \qquad \textit{for every } x, z \in X\,, \tag{2.65}$$

and

$$\big\langle x, I \cdot x \big\rangle_{X \times X^*} > 0 \qquad \forall x \neq 0 \in X\,. \tag{2.66}$$

Then there exists a Hilbert space H and an injective operator $T \in \mathcal{L}(X, H)$ (i.e. $\ker T = \{0\}$), whose image is dense in H, and such that if we consider the operator $\widetilde{T} \in \mathcal{L}(H; X^*)$, defined by the formula (2.62)*, then we will have*

$$(\widetilde{T} \circ T) \cdot x = I \cdot x \qquad \forall x \in X\,. \tag{2.67}$$

I.e. $\{X, H, X^*\}$ is an evolution triple with the corresponding inclusion operator $T \in \mathcal{L}(X; H)$, as it was defined in Definition 2.9, together with the corresponding operator $\widetilde{T} \in \mathcal{L}(H; X^*)$, defined as in (2.62)*, and $I \equiv \widetilde{T} \circ T$.*

Proof. Since I is a self-adjoint and strictly positive operator, the identity

$$\big\langle\big\langle x, z \big\rangle\big\rangle := \big\langle x, I \cdot z \big\rangle_{X \times X^*} = \big\langle z, I \cdot x \big\rangle_{X \times X^*} \qquad \text{for every } x, z \in X\,, \tag{2.68}$$

defines a scalar product in X and the corresponding Euclidian norm $\big|\big|x\big|\big| := \sqrt{<< x, x >>}$. Denote the closure of the space X with respect to this Euclidian norm by H and the trivial embedding of X into H by T. Thus H will be a Hilbert space and $T \in \mathcal{L}(X, H)$ will be an injective bounded linear operator whose image is dense in H. Moreover,

$$\big\langle T \cdot x, T \cdot z \big\rangle_{H \times H} = \big\langle\big\langle x, z \big\rangle\big\rangle = \big\langle x, I \cdot z \big\rangle_{X \times X^*} \qquad \text{for every } x, z \in X\,. \tag{2.69}$$

Thus if we consider the operator $\widetilde{T} \in \mathcal{L}(H; X^*)$, defined as in (2.62), by the formula

$$\big\langle x, \widetilde{T} \cdot y \big\rangle_{X \times X^*} := \big\langle T \cdot x, y \big\rangle_{H \times H} \qquad \text{for every } y \in H \text{ and } x \in X\,, \tag{2.70}$$

then plugging (2.70) into (2.69) we deduce

$$\big\langle x, (\widetilde{T} \circ T) \cdot z \big\rangle_{X \times X^*} = \big\langle T \cdot x, T \cdot z \big\rangle_{H \times H} = \big\langle x, I \cdot z \big\rangle_{X \times X^*} \qquad \text{for every } x, z \in X\,. \tag{2.71}$$

I.e. $\widetilde{T} \circ T \equiv I$. □

Next as a particular case of Lemma 2.6 we have the following Corollary.

Corollary 2.1. *Let $\{X,H,X^*\}$ be an evolution triple with the corresponding inclusion operator $T\in\mathcal{L}(X;H)$, as it was defined in Definition 2.9, together with the corresponding operator $\widetilde{T}\in\mathcal{L}(H;X^*)$, defined as in (2.62), and let $a,b\in\mathbb{R}$ be s.t. $a<b$. Let $w(t)\in L^\infty(a,b;H)$ be such that the function $v:[a,b]\to X^*$ defined by $v(t):=\widetilde{T}\cdot\big(w(t)\big)$ belongs to $W^{1,q}(a,b;X^*)$ for some $q\geq 1$. Then we can redefine w on a subset of $[a,b]$ of Lebesgue measure zero, so that $w(t)$ will be H-weakly continuous in t on $[a,b]$ (i.e. $w\in C^0_w(a,b;H)$). Moreover, for every $a\leq\alpha<\beta\leq b$ and for every $\delta(t)\in C^1\big([a,b];X\big)$ we will have*

$$\int_\alpha^\beta\bigg\{\Big\langle\delta(t),\frac{dv}{dt}(t)\Big\rangle_{X\times X^*}+\Big\langle\frac{d\delta}{dt}(t),v(t)\Big\rangle_{X\times X^*}\bigg\}dt=\big\langle T\cdot\delta(\beta),w(\beta)\big\rangle_{H\times H}-\big\langle T\cdot\delta(\alpha),w(\alpha)\big\rangle_{H\times H}\,. \tag{2.72}$$

Lemma 2.8. *Let $\{X,H,X^*\}$ be an evolution triple with the corresponding inclusion operator $T\in\mathcal{L}(X;H)$, as it was defined in Definition 2.9, together with the corresponding operator $\widetilde{T}\in\mathcal{L}(H;X^*)$, defined as in (2.62), and let $a,b\in\mathbb{R}$ be s.t. $a<b$. Let $u(t)\in L^q(a,b;X)$ for some $q>1$ such that the function $v(t):[a,b]\to X^*$ defined by $v(t):=I\cdot\big(u(t)\big)$ belongs to $W^{1,q^*}(a,b;X^*)$ for $q^*:=q/(q-1)$, where we denote $I:=\widetilde{T}\circ T:X\to X^*$. Then the function $w(t):[a,b]\to H$ defined by $w(t):=T\cdot\big(u(t)\big)$ belongs to $L^\infty(a,b;H)$ and for every subinterval $[\alpha,\beta]\subset[a,b]$ we have*

$$\int_\alpha^\beta\Big\langle u(t),\frac{dv}{dt}(t)\Big\rangle_{X\times X^*}dt=\frac12\Big(\|w(\beta)\|_H^2-\|w(\alpha)\|_H^2\Big)\,, \tag{2.73}$$

up to a redefinition of $w(t)$ on a subset of $[a,b]$ of Lebesgue measure zero, such that w is H-weakly continuous, as it was stated in Corollary 2.1.

Proof. Clearly for every $t\in[a,b]$ we have $\widetilde{T}\cdot w(t)=v(t)$. Next by (2.5), for every $\delta(t)\in C^1\big([a,b];X\big)$ and every subinterval $[\alpha,\beta]\subset[a,b]$ we have

$$\int_\alpha^\beta\bigg\{\Big\langle\delta(t),\frac{dv}{dt}(t)\Big\rangle_{X\times X^*}+\Big\langle\frac{d\delta}{dt}(t),v(t)\Big\rangle_{X\times X^*}\bigg\}dt=\big\langle\delta(\beta),v(\beta)\big\rangle_{X\times X^*}-\big\langle\delta(\alpha),v(\alpha)\big\rangle_{X\times X^*}\,. \tag{2.74}$$

Fix two numbers $a\leq\alpha<\beta\leq b$. Let $\eta\in C_c^\infty(\mathbb{R},\mathbb{R})$ be a mollifying kernel, satisfying $\eta\geq 0$, $\int_{\mathbb{R}}\eta(t)dt=1$, $\operatorname{supp}\eta\subset[-1,1]$ and $\eta(-t)=\eta(t)$ $\forall t\in\mathbb{R}$. For every $0<\varepsilon<(\beta-\alpha)/2$ and every $t\in[a,b]$ consider $u_\varepsilon(t)\in X$, $w_\varepsilon(t)\in H$ and $v_\varepsilon(t)\in X^*$ defined by

$$\big\langle u_\varepsilon(t),\delta\big\rangle_{X\times X^*}:=\frac1\varepsilon\int_\alpha^\beta\eta\Big(\frac{s-t}{\varepsilon}\Big)\big\langle u(s),\delta\big\rangle_{X\times X^*}ds\qquad\forall\delta\in X^*\,, \tag{2.75}$$

$$\big\langle w_\varepsilon(t),h\big\rangle_{H\times H}:=\frac1\varepsilon\int_\alpha^\beta\eta\Big(\frac{s-t}{\varepsilon}\Big)\big\langle w(s),h\big\rangle_{H\times H}ds\qquad\forall h\in H\,, \tag{2.76}$$

$$\big\langle x,v_\varepsilon(t)\big\rangle_{X\times X^*}:=\frac1\varepsilon\int_\alpha^\beta\eta\Big(\frac{s-t}{\varepsilon}\Big)\big\langle x,v(s)\big\rangle_{X\times X^*}ds\qquad\forall x\in X\,, \tag{2.77}$$

(It is clear that the formulas (2.75), (2.76) and (2.77) define bounded linear functionals on X^*, H and X respectively and thus the elements of X, H and X^*). We also easily obtain

$$T\cdot\big(u_\varepsilon(t)\big)=w_\varepsilon(t)\quad\text{and}\quad\widetilde{T}\cdot\big(w_\varepsilon(t)\big)=v_\varepsilon(t)\,. \tag{2.78}$$

Moreover, clearly $u_\varepsilon(t)\in C^1([a,b],X)$, $v_\varepsilon(t)\in C^1([a,b],X^*)$ and

$$\Big\langle\frac{du_\varepsilon}{dt}(t),\delta\Big\rangle_{X\times X^*}=-\frac{1}{\varepsilon^2}\int_\alpha^\beta\eta'\Big(\frac{s-t}{\varepsilon}\Big)\big\langle u(s),\delta\big\rangle_{X\times X^*}ds\qquad\forall\delta\in X^*\,, \tag{2.79}$$

$$\Big\langle x,\frac{dv_\varepsilon}{dt}(t)\Big\rangle_{X\times X^*}=-\frac{1}{\varepsilon^2}\int_\alpha^\beta\eta'\Big(\frac{s-t}{\varepsilon}\Big)\big\langle x,v(s)\big\rangle_{X\times X^*}ds\qquad\forall x\in X\,. \tag{2.80}$$

Next since $u(t) \in L^q(a,b;X)$ and $v(t) \in L^{q^*}(a,b;X^*)$, by (2.3) from Lemma 2.1, we infer

$$\lim_{\varepsilon\to 0^+} u_\varepsilon(t) = u(t) \quad \text{in } L^q(\alpha,\beta;X) \quad \text{and} \quad \lim_{\varepsilon\to 0^+} v_\varepsilon(t) = v(t) \quad \text{in } L^{q^*}(\alpha,\beta;X^*)\,. \tag{2.81}$$

Then, testing (2.74) with $u_\varepsilon(t)$ instead of $\delta(t)$, we obtain

$$\int_\alpha^\beta \Big\langle \frac{du_\varepsilon}{dt}(t), v(t)\Big\rangle_{X\times X^*} dt = -\int_\alpha^\beta \Big\langle u_\varepsilon(t), \frac{dv}{dt}(t)\Big\rangle_{X\times X^*} dt + \big\langle u_\varepsilon(\beta), v(\beta)\big\rangle_{X\times X^*} - \big\langle u_\varepsilon(\alpha), v(\alpha)\big\rangle_{X\times X^*}\,. \tag{2.82}$$

On the other hand, using (2.79) we obtain

$$\int_\alpha^\beta \Big\langle \frac{du_\varepsilon}{dt}(t), v(t)\Big\rangle_{X\times X^*} dt = -\frac{1}{\varepsilon^2}\int_\alpha^\beta\int_\alpha^\beta \eta'\Big(\frac{s-t}{\varepsilon}\Big)\big\langle u(s), v(t)\big\rangle_{X\times X^*} ds dt =$$
$$-\frac{1}{\varepsilon^2}\int_\alpha^\beta\int_\alpha^\beta \eta'\Big(\frac{s-t}{\varepsilon}\Big)\Big\langle u(s), (\widetilde{T}\circ T)\cdot\big(u(t)\big)\Big\rangle_{X\times X^*} ds dt =$$
$$-\frac{1}{\varepsilon^2}\int_\alpha^\beta\int_\alpha^\beta \eta'\Big(\frac{s-t}{\varepsilon}\Big)\Big\langle T\cdot\big(u(s)\big), T\cdot\big(u(t)\big)\Big\rangle_{H\times H} ds dt\,. \tag{2.83}$$

However, since η is even and therefore η' is odd, we have

$$\frac{2}{\varepsilon^2}\int_\alpha^\beta\int_\alpha^\beta \eta'\Big(\frac{s-t}{\varepsilon}\Big)\Big\langle T\cdot\big(u(s)\big), T\cdot\big(u(t)\big)\Big\rangle_{H\times H} ds dt =$$
$$\frac{1}{\varepsilon^2}\int_\alpha^\beta\int_\alpha^\beta \eta'\Big(\frac{s-t}{\varepsilon}\Big)\Big\langle T\cdot\big(u(s)\big), T\cdot\big(u(t)\big)\Big\rangle_{H\times H} ds dt + \frac{1}{\varepsilon^2}\int_\alpha^\beta\int_\alpha^\beta \eta'\Big(\frac{t-s}{\varepsilon}\Big)\Big\langle T\cdot\big(u(t)\big), T\cdot\big(u(s)\big)\Big\rangle_{H\times H} dt ds =$$
$$\frac{1}{\varepsilon^2}\int_\alpha^\beta\int_\alpha^\beta \eta'\Big(\frac{s-t}{\varepsilon}\Big)\Big\langle T\cdot\big(u(s)\big), T\cdot\big(u(t)\big)\Big\rangle_{H\times H} ds dt - \frac{1}{\varepsilon^2}\int_\alpha^\beta\int_\alpha^\beta \eta'\Big(\frac{s-t}{\varepsilon}\Big)\Big\langle T\cdot\big(u(s)\big), T\cdot\big(u(t)\big)\Big\rangle_{H\times H} ds dt = 0\,.$$

Therefore, by (2.83) we deduce

$$\int_\alpha^\beta \Big\langle \frac{du_\varepsilon}{dt}(t), v(t)\Big\rangle_{X\times X^*} dt = 0\,. \tag{2.84}$$

Plugging it to (2.82) and using (2.81) we obtain

$$\lim_{\varepsilon\to 0^+}\Big\{\big\langle u_\varepsilon(\beta), v(\beta)\big\rangle_{X\times X^*} - \big\langle u_\varepsilon(\alpha), v(\alpha)\big\rangle_{X\times X^*}\Big\} = \int_\alpha^\beta \Big\langle u(t), \frac{dv}{dt}(t)\Big\rangle_{X\times X^*} dt\,. \tag{2.85}$$

On the other hand, using (2.4) from Lemma 2.1, we infer that there exists a sequence $\varepsilon_n \to 0^+$, such that for a.e. $\alpha,\beta \in (a,b)$ we have

$$\big\langle u_{\varepsilon_n}(\alpha), v(\alpha)\big\rangle_{X\times X^*} = \frac{1}{\varepsilon_n}\int_\alpha^\beta \eta\Big(\frac{\tau-\alpha}{\varepsilon_n}\Big)\big\langle u(\tau), v(\alpha)\big\rangle_{X\times X^*} d\tau$$
$$= \int_0^1 \eta(s)\big\langle u(\alpha+\varepsilon_n s), v(\alpha)\big\rangle_{X\times X^*} ds \to \frac{1}{2}\big\langle u(\alpha), v(\alpha)\big\rangle_{X\times X^*}\,, \tag{2.86}$$

and by the same way for a.e. $\alpha, \beta \in (a,b)$ we have

$$\big\langle u_{\varepsilon_n}(\beta), v(\beta)\big\rangle_{X\times X^*} \to \frac{1}{2}\big\langle u(\beta), v(\beta)\big\rangle_{X\times X^*}. \tag{2.87}$$

(Note that u_ε in (2.86) and (2.87) depends on the interval (α,β)). Thus, using (2.85) together with (2.86) and (2.87), for a.e. α, β we obtain

$$\frac{1}{2}\big\langle u(\beta), v(\beta)\big\rangle_{X\times X^*} - \frac{1}{2}\big\langle u(\alpha), v(\alpha)\big\rangle_{X\times X^*} = \int\limits_\alpha^\beta \Big\langle u(t), \frac{dv}{dt}(t)\Big\rangle_{X\times X^*} dt. \tag{2.88}$$

Therefore, using (2.88), for a.e. α, β we deduce

$$\frac{1}{2}\|w(\beta)\|_H^2 - \frac{1}{2}\|w(\alpha)\|_H^2 = \int\limits_\alpha^\beta \Big\langle u(t), \frac{dv}{dt}(t)\Big\rangle_{X\times X^*} dt. \tag{2.89}$$

In particular we obtain $w \in L^\infty(a,b;H)$. Thus we can redefine w on a subset of $[a,b]$ of Lebesgue measure zero, so that $w(t)$ will be H-weakly continuous, as it was stated in Corollary 2.1. Finally by (2.85) for all $[\alpha,\beta] \subset [a,b]$ we have

$$\lim_{\varepsilon\to 0^+}\Big\{\big\langle w_\varepsilon(\beta), w(\beta)\big\rangle_{H\times H} - \big\langle w_\varepsilon(\alpha), w(\alpha)\big\rangle_{H\times H}\Big\} = \int\limits_\alpha^\beta \Big\langle u(t), \frac{dv}{dt}(t)\Big\rangle_{X\times X^*} dt. \tag{2.90}$$

However, since w is H-weakly continuous for all α, β, as $\varepsilon \to 0^+$ we have

$$\begin{aligned}\big\langle w_\varepsilon(\alpha), w(\alpha)\big\rangle_{H\times H} &= \frac{1}{\varepsilon}\int\limits_\alpha^\beta \eta\Big(\frac{\tau-\alpha}{\varepsilon}\Big)\big\langle w(\tau), w(\alpha)\big\rangle_{H\times H} d\tau \\ &= \int\limits_0^1 \eta(s)\big\langle w(\alpha+\varepsilon s), w(\alpha)\big\rangle_{H\times H} ds \to \frac{1}{2}\|w(\alpha)\|_H^2, \quad \text{and} \\ \big\langle w_\varepsilon(\beta), w(\beta)\big\rangle_{H\times H} &= \frac{1}{\varepsilon}\int\limits_\alpha^\beta \eta\Big(\frac{\tau-\beta}{\varepsilon}\Big)\big\langle w(\tau), w(\beta)\big\rangle_{H\times H} d\tau \\ &= \int\limits_{-1}^0 \eta(s)\big\langle w(\beta+\varepsilon s), w(\beta)\big\rangle_{H\times H} ds \to \frac{1}{2}\|w(\beta)\|_H^2.\end{aligned}$$

Therefore, by (2.90) we obtain (2.73) already for all α, β. □

We will need in the sequel the following compactness results.

Lemma 2.9. *Let X, Y Z be three Banach spaces, such that X is a reflexive space. Furthermore, let $T \in \mathcal{L}(X;Y)$ and $S \in \mathcal{L}(X;Z)$ be bounded linear operators. Moreover assume that S is an injective inclusion (i.e. it satisfies $\ker S = \{0\}$) and T is a compact operator. Assume that $a, b \in \mathbb{R}$ such that $a < b$, $1 \le q < +\infty$ and $\{u_n(t)\} \subset L^q(a,b;X)$ is a bounded in $L^q(a,b;X)$ sequence of functions, such that the functions $v_n(t) : (a,b) \to Z$, defined by $v_n(t) := S \cdot \big(u_n(t)\big)$, belongs to $L^\infty(a,b;Z)$, the sequence $\{v_n(t)\}$ is bounded in $L^\infty(a,b;Z)$ and for a.e. $t \in (a,b)$ we have*

$$v_n(t) \rightharpoonup v(t) \quad \textit{weakly in } Z \textit{ as } n \to +\infty. \tag{2.91}$$

Then,

$$\big\{T \cdot \big(u_n(t)\big)\big\} \quad \textit{converges strongly in } L^q(a,b;Y). \tag{2.92}$$

Proof. First of all we would like to observe that without loss of generality we may assume that the spaces X, Y and Z are separable. Indeed, in the general case since $u_n(t) \in L^q(a,b;X)$ then u_n is strictly measurable. Thus in particular u_n is separately valued, i.e. for every n there exists a separable subspace $X_n \subset X$ such that $u_n(t) \in X_n$ for a.e. $t \in (a,b)$. Define $\bar{X}$ be the closure in X of the linear span of $\bigcup_{n=1}^{+\infty} X_n$. Then $\bar{X}$ is a separable subspace of X by itself and by the construction for a.e. $t \in (a,b)$ for every n we have $u_n(t) \in \bar{X}$. Then we also can define $\bar{Y}$ and $\bar{Z}$ as the closures of the images of the subspace $\bar{X}$ under the transformations T and S respectively. So $\bar{Y}$ and $\bar{Z}$ are separable subspaces of Y and Z respectively. The new spaces $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ together with the operators $T \llcorner \bar{X}$ and $S \llcorner \bar{X}$ and the functions $u_n(t) \in L^q(a,b;\bar{X})$ satisfy all the conditions of the present lemma.

Therefore, from now we assume the spaces X, Y and Z to be separable. Thus by Lemma A.3 from the Appendix there exists a separable Hilbert space U and an operator $L \in \mathcal{L}(Z;U)$ such that L is injective (i.e. $\ker L = \{0\}$) and compact. Thus since L is a compact operator, by (2.91), for a.e. $t \in (a,b)$ we have

$$L \cdot v_n(t) \to L \cdot v(t) \quad \text{strongly in } U \text{ as } n \to +\infty \,. \tag{2.93}$$

Moreover, the sequence $\{L \cdot v_n(t)\}$ is bounded in $L^\infty(a,b;U)$. Therefore, by the Dominated Convergence Theorem we deduce that

$$\big(L \circ S\big) \cdot \big(u_n(t)\big) = L \cdot \big(v_n(t)\big) \to L \cdot \big(v(t)\big) \quad \text{strongly in } L^q(a,b;U) \text{ as } n \to +\infty \,. \tag{2.94}$$

Next since S and L are injective inclusions, we deduce that $L \circ S \in \mathcal{L}(X,U)$ is an injective inclusion. Therefore, using Lemma A.1 from the Appendix we deduce that for every $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that for all $n, m \in \mathbb{N}$ we must have

$$\Big\|T\cdot u_{n+m}(t) - T\cdot u_n(t)\Big\|_Y \leq \varepsilon\Big\|u_{n+m}(t) - u_n(t)\Big\|_X + c_\varepsilon\Big\|(L\circ S)\cdot u_{n+m}(t) - (L\circ S)\cdot u_n(t)\Big\|_U \quad \forall t \in (a,b)\,. \tag{2.95}$$

Therefore, for every $\varepsilon > 0$, for all $n, m \in \mathbb{N}$ we obtain

$$\begin{aligned}\Big\|T \cdot \big(u_{n+m}(t)\big) - T \cdot \big(u_n(t)\big)\Big\|_{L^q(a,b;Y)} \\ \leq \varepsilon\Big\|u_{n+m}(t) - u_n(t)\Big\|_{L^q(a,b;X)} + c_\varepsilon\Big\|L \cdot \big(v_{n+m}(t)\big) - L \cdot \big(v_n(t)\big)\Big\|_{L^q(a,b;U)} \,.\end{aligned} \tag{2.96}$$

However, since the sequence $\{u_n\}$ is bounded in $L^q(a,b;X)$, using (2.96) we deduce that there exists a constant $C_0 > 0$ independent on ε such that

$$\Big\|T \cdot \big(u_{n+m}(t)\big) - T \cdot \big(u_n(t)\big)\Big\|_{L^q(a,b;Y)} \leq C_0\varepsilon + c_\varepsilon\Big\|L \cdot \big(v_{n+m}(t)\big) - L \cdot \big(v_n(t)\big)\Big\|_{L^q(a,b;U)} \,. \tag{2.97}$$

On the other hand, by (2.94), there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that for every $n > n_0$ and every $m \in \mathbb{N}$ we have

$$\Big\|L \cdot \big(v_{n+m}(t)\big) - L \cdot \big(v_n(t)\big)\Big\|_{L^q(a,b;U)} \leq \frac{\varepsilon}{c_\varepsilon} \,. \tag{2.98}$$

Thus, plugging (2.98) into (2.97), we obtain that for every $n > n_0$ and every $m \in \mathbb{N}$ we must have

$$\Big\|T \cdot \big(u_{n+m}(t)\big) - T \cdot \big(u_n(t)\big)\Big\|_{L^q(a,b;Y)} \leq (C_0 + 1)\varepsilon \,. \tag{2.99}$$

Therefore, since $\varepsilon > 0$ in (2.99) is arbitrary and since $L^q(a,b;Y)$ is a Banach space we finally deduce (2.92). $\square$

Lemma 2.10. *Let Z be a reflexive Banach space and let $\big\{v_n(t)\big\}_{n=1}^{+\infty} \subset W^{1,1}(a,b;Z)$ be a sequence of functions, bounded in $W^{1,1}(a,b;Z)$. Then, $\big\{v_n(t)\big\}_{n=1}^{+\infty}$ is bounded in $L^\infty(a,b;Z)$ and, up to a subsequence, we have*

$$v_n(t) \rightharpoonup v(t) \quad \textit{weakly in } Z \textit{ as } \; n \to +\infty \,, \quad \textit{for a.e } t \in (a,b)\,. \tag{2.100}$$

Proof. As before without loss of generality we may assume that the space Z is separable. Thus since Z is a reflexive we deduce that the space Z^* is also separable. Next since $W^{1,1}(a,b;Z)$ is continuously embedded in $L^\infty(a,b;Z)$, there exists a constant $C>0$ such that

$$\big\|v_n(t)\big\|_Z \le C \quad \forall t \in \mathfrak{R}_0 \subset (a,b)\,, \tag{2.101}$$

where $\mathcal{L}^1\big((a,b)\setminus\mathfrak{R}_0\big)=0$. On the other hand, since Z^* is separable, there exists a sequence $\{\sigma_j\}_{j=1}^{+\infty}\subset Z^*$ which is dense in Z^*. For every $j,n\in\mathbb{N}$ and every $t\in(a,b)$ set

$$h_n^{(j)}(t) := \big\langle v_n(t),\sigma_j\big\rangle_{Z\times Z^*} \in W^{1,1}(a,b;\mathbb{R})\,. \tag{2.102}$$

Then for every fixed j the sequence $\{h_n^{(j)}(t)\}_{n=1}^{+\infty}$ is bounded in $W^{1,1}(a,b;\mathbb{R})$. Thus since the embedding of $W^{1,1}(a,b;\mathbb{R})$ into $L^1(a,b;\mathbb{R})$ is compact and since every L^1-convergent sequence has a subsequence which converges almost everywhere, there exists an increasing sequence $\{n_k^{(1)}\}_{k=1}^{+\infty}\subset\mathbb{N}$ and a set $\mathfrak{R}_1\subset\mathfrak{R}_0$ such that $\mathcal{L}^1\big((a,b)\setminus\mathfrak{R}_1\big)=0$ and $\lim_{k\to+\infty} h^{(1)}_{n_k^{(1)}}(t)=\bar h^{(1)}(t)$ for every $t\in\mathfrak{R}_1$. In the same way there exists a further subsequence $\{n_k^{(2)}\}_{k=1}^{+\infty}\subset\{n_k^{(1)}\}_{k=1}^{+\infty}$ and a set $\mathfrak{R}_2\subset\mathfrak{R}_1$ such that $\mathcal{L}^1\big((a,b)\setminus\mathfrak{R}_2\big)=0$ and $\lim_{k\to+\infty} h^{(2)}_{n_k^{(2)}}(t)=\bar h^{(2)}(t)$ for every $t\in\mathfrak{R}_2$. Continuing this process we obtain that for every $m\in\mathbb{N}$ there exists a subsequence $\{n_k^{(m+1)}\}_{k=1}^{+\infty}\subset\{n_k^{(m)}\}_{k=1}^{+\infty}$ and a set $\mathfrak{R}_{m+1}\subset\mathfrak{R}_m$ such that $\mathcal{L}^1\big((a,b)\setminus\mathfrak{R}_{m+1}\big)=0$ and $\lim_{k\to+\infty} h^{(m+1)}_{n_k^{(m+1)}}(t)=\bar h^{(m+1)}(t)$ for every $t\in\mathfrak{R}_{m+1}$. Thus taking the diagonal subsequence we obtain that, up to a subsequence, still denoted by $v_n(t)$ we have

$$\lim_{n\to+\infty}\big\langle v_n(t),\sigma_j\big\rangle_{Z\times Z^*}=\bar h^{(j)}(t)\quad \forall t\in\bar{\mathfrak{R}},\ \ \forall j\in\mathbb{N}\,,$$

where $\bar{\mathfrak{R}}:=\cap_{j=1}^{+\infty}\mathfrak{R}_j$. Moreover, clearly we have $\mathcal{L}^1\big((a,b)\setminus\bar{\mathfrak{R}}\big)=0$. Therefore by the fact that $\{\sigma_j\}_{j=1}^{+\infty}\subset Z^*$ is dense in Z^* and by (2.101) we obtain that

$$v_n(t)\rightharpoonup v(t)\quad \text{weakly in } Z \text{ as } n\to+\infty\,,\quad \forall t\in\bar{\mathfrak{R}}\,,$$

i.e. almost everywhere in (a,b). □

As a direct consequence of Lemma 2.9 and Lemma 2.10 we have the following Lemma.

Lemma 2.11. *Let X, Y and Z be three Banach spaces, such that X and Z are reflexive. Furthermore, let $T\in\mathcal{L}(X;Y)$ and $S\in\mathcal{L}(X;Z)$ be bounded linear operators. Moreover assume that S is an injective inclusion (i.e. it satisfies $\ker S=\{0\}$) and T is a compact operator. Assume that $a,b\in\mathbb{R}$ such that $a<b$, $1\le q<+\infty$ and $\{u_n(t)\}\subset L^q(a,b;X)$ is a bounded in $L^q(a,b;X)$ sequence of functions, such that the functions $v_n(t):(a,b)\to Z$, defined by $v_n(t):=S\cdot\big(u_n(t)\big)$, belongs to $W^{1,1}(a,b;Z)$ and the sequence $\big\{\frac{dv_n}{dt}(t)\big\}$ is bounded in $L^1(a,b;Z)$. Then, up to a subsequence,*

$$\big\{T\cdot\big(u_n(t)\big)\big\}\quad \textit{converges strongly in } L^q(a,b;Y)\,. \tag{2.103}$$

3 The properties of the Euler-Lagrange equations

Definition 3.1. Let $\{X,H,X^*\}$ be an evolution triple with the corresponding inclusion operator $T\in\mathcal{L}(X;H)$, as it was defined in Definition 2.9, together with the corresponding operator $\widetilde{T}\in\mathcal{L}(H;X^*)$, defined as in (2.62), and let $a,b\in\mathbb{R}$ be s.t. $a<b$. Let $u(t)\in L^q(a,b;X)$ for some $q>1$ such that the function $v(t):[a,b]\to X^*$ defined by $v(t):=I\cdot\big(u(t)\big)$ belongs to $W^{1,q^*}(a,b;X^*)$ for $q^*:=q/(q-1)$, where $I:=\widetilde{T}\circ T:X\to X^*$. Denote the set of all such functions u by $\mathcal{R}_q(a,b)$. Note that by Lemma 2.8, for every $u(t)\in\mathcal{R}_q(a,b)$ the function $w(t):[a,b]\to H$ defined by $w(t):=T\cdot\big(u(t)\big)$ belongs to $L^\infty(a,b;H)$ and, up to a redefinition of $w(t)$ on a subset of $[a,b]$ of Lebesgue measure zero, w is H-weakly continuous, as it was stated in Corollary 2.1.

Definition 3.2. Let $\{X, H, X^*\}$ be an evolution triple with the corresponding inclusion operator $T \in \mathcal{L}(X; H)$, as it was defined in Definition 2.9, together with the corresponding operator $\widetilde{T} \in \mathcal{L}(H; X^*)$, defined as in (2.62), and let $\lambda \in \{0, 1\}$ and $a, b, q \in \mathbb{R}$ be s.t. $a < b$ and $q \geq 2$. Furthermore, for every $t \in [a, b]$ let $\Psi_t(x) : X \to [0, +\infty)$ be a strictly convex function which is Gateaux differentiable at every $x \in X$, satisfies $\Psi_t(0) = 0$ and satisfies the growth condition

$$(1/C_0)\,\|x\|_X^q - C_0 \leq \Psi_t(x) \leq C_0\,\|x\|_X^q + C_0 \quad \forall x \in X,\ \forall t \in [a, b]\,, \tag{3.1}$$

for some $C_0 > 0$. We also assume that $\Psi_t(x)$ is Borel on the pair of variables (x, t) (see Definition 2.2). For every $t \in [a, b]$ denote by Ψ_t^* the Legendre transform of Ψ_t, defined by

$$\Psi_t^*(y) := \sup\big\{ < z, y >_{X \times X^*} - \Psi_t(z) :\ z \in X \big\} \qquad \forall y \in X^*\,.$$

Next for every $t \in [a, b]$ let $\Lambda_t(x) : X \to X^*$ be a function which is Gateaux differentiable at every $x \in X$, $\Lambda_t(0) \in L^{q^*}(a, b; X^*)$ and the derivative of Λ_t satisfies the growth condition

$$\|D\Lambda_t(x)\|_{\mathcal{L}(X;X^*)} \leq g\big(\|T \cdot x\|_H\big)\left(\|x\|_X^{q-2} + 1\right) \quad \forall x \in X,\ \forall t \in [a, b]\,, \tag{3.2}$$

for some non-decreasing function $g(s) : [0 + \infty) \to (0, +\infty)$. We also assume that $\Lambda_t(x)$ is strongly Borel on the pair of variables (x, t) (see Definition 2.2). Assume also that Ψ_t and Λ_t satisfy the following monotonicity condition

$$\Big\langle h, \lambda\Big\{D\Psi_t\big(\lambda x + h\big) - D\Psi_t\big(\lambda x\big)\Big\} + D\Lambda_t(x) \cdot h\Big\rangle_{X \times X^*} \geq -\hat{g}\big(\|T \cdot x\|_H\big)\Big(\|x\|_X^q + \mu(t)\Big)\,\|T \cdot h\|_H^2$$
$$\forall x, h \in X,\ \forall t \in [a, b]\,, \tag{3.3}$$

for some non-decreasing function $\hat{g}(s) : [0 + \infty) \to (0, +\infty)$ and some nonnegative function $\mu(t) \in L^1(a, b; \mathbb{R})$. Define the functional $J(u) : \mathcal{R}_q(a, b) \to \mathbb{R}$ (where $\mathcal{R}_q(a, b)$ was defined in Definition 3.1) by

$$J(u) :=$$
$$\int_a^b \left\{\Psi_t\big(\lambda u(t)\big) + \Psi_t^*\left(-\frac{dv}{dt}(t) - \Lambda_t\big(u(t)\big)\right) + \lambda\Big\langle u(t), \Lambda_t\big(u(t)\big)\Big\rangle_{X \times X^*}\right\}dt + \frac{\lambda}{2}\Big(\|w(b)\|_H^2 - \|w(a)\|_H^2\Big)\,, \tag{3.4}$$

where $w(t) := T \cdot \big(u(t)\big)$, $v(t) := I \cdot \big(u(t)\big) = \widetilde{T} \cdot \big(w(t)\big)$ with $I := \widetilde{T} \circ T :\ X \to X^*$ and we assume that $w(t)$ is H-weakly continuous on $[a, b]$, as it was stated in Corollary 2.1. Finally, for every $w_0 \in H$ consider the minimization problem

$$\inf\Big\{J(u) :\ u \in \mathcal{R}_q(a, b),\ w(a) = w_0\Big\}\,. \tag{3.5}$$

Remark 3.1. Note that by Lemma 2.3, for every $t \in [a, b]$ $\Psi_t^*(y)$ is a strictly convex function from X^* to $[0, +\infty)$, satisfies $\Psi_t^*(0) = 0$ and

$$(1/C)\,\|y\|_{X^*}^{q^*} - C \leq \Psi_t^*(y) \leq C\,\|y\|_{X^*}^{q^*} + C \quad \forall y \in X^*\ \forall t \in [a, b]\,, \tag{3.6}$$

for some $C > 0$ where $q^* := q/(q-1)$. Moreover, $\Psi_t^*(y)$ is Gateaux differentiable at every $y \in X^*$, and $x \in X$ satisfies $x = D\Psi_t^*(y)$ if and only if $y \in X^*$ satisfies $y = D\Psi_t(x)$. Note also that $\Psi_t^*(y)$ is Borel mapping on the pair of variables (y, t). Finally note that since by Lemma 2.8 we have

$$\int_a^b \Big\langle u(t), \frac{dv}{dt}(t)\Big\rangle_{X \times X^*} dt = \frac{1}{2}\Big(\|w(b)\|_H^2 - \|w(a)\|_H^2\Big)\,,$$

we can rewrite the definition of J in (3.4) by

$$J(u) := \int_a^b \left\{\Psi_t\big(\lambda u(t)\big) + \Psi_t^*\left(-\frac{dv}{dt}(t) - \Lambda_t\big(u(t)\big)\right) + \lambda\Big\langle u(t), \frac{dv}{dt}(t) + \Lambda_t\big(u(t)\big)\Big\rangle_{X \times X^*}\right\}dt\,. \tag{3.7}$$

Then by the definition of the Legendre transform we deduce that $J(u) \geq 0$ for every $u \in \mathcal{R}_q(a,b)$ and $J(u) = 0$ if and only if $u(t)$ is a solution of

$$\frac{dv}{dt}(t) + \Lambda_t\big(u(t)\big) + D\Psi_t\big(\lambda u(t)\big) = 0 \quad \text{for a.e. } t \in (a,b)\,. \tag{3.8}$$

Remark 3.2. Assume that, instead of (3.3), one requires that Ψ_t and Λ_t satisfy the following inequality

$$\Big\langle h, \lambda\Big\{D\Psi_t\big(\lambda x + h\big) - D\Psi_t(\lambda x)\Big\} + D\Lambda_t(x)\cdot h\Big\rangle_{X\times X^*} \geq \\ \frac{\|h\|_X^2}{\tilde{g}(\|T\cdot x\|_H)} - \tilde{g}\big(\|T\cdot x\|_H\big)\Big(\|x\|_X^q + \mu(t)\Big)^{(2-r)/2}\|h\|_X^r\,\|T\cdot h\|_H^{(2-r)} \quad \forall x,h\in X,\ \forall t\in[a,b], \tag{3.9}$$

for some non-decreasing function $\tilde{g}(s) : [0+\infty) \to (0,+\infty)$, some nonnegative function $\mu(t) \in L^1(a,b;\mathbb{R})$ and some constant $r \in (0,2)$. Then, (3.3) follows by the trivial inequality $(r/2)\,a^2 + \big((2-r)/2\big)\,b^2 \geq a^r\,b^{2-r}$, valid for every two nonnegative real numbers a and b.

Lemma 3.1. *Let $\{X,H,X^*\}$ be an evolution triple with the corresponding inclusion operator $T \in \mathcal{L}(X;H)$, as it was defined in Definition 2.9, together with the corresponding operator $\widetilde{T} \in \mathcal{L}(H;X^*)$, defined as in (2.62). Furthermore, Let a,b,q,λ be such that $a<b$, $q\geq 2$ and $\lambda\in\{0,1\}$. Assume that Ψ_t and Λ_t satisfy all the conditions of Definition 3.2. Furthermore, let $\mathcal{R}_q(a,b)$ and J be as in Definitions 3.1 and 3.2 respectively. Then for every $u\in\mathcal{R}_q(a,b)$ we have $J(u)<+\infty$. Moreover, for every $u,h\in\mathcal{R}_q(a,b)$ and every $s\in\mathbb{R}$ we have $(u+sh)\in\mathcal{R}_q(a,b)$ and*

$$\lim_{s\to 0}\frac{1}{s}\Big(J(u+sh)-J(u)\Big) = \int_a^b\Bigg\{\Big\langle h(t), \lambda\Big\{D\Psi_t\big(\lambda u(t)\big) - H_u(t)\Big\}\Big\rangle_{X\times X^*} \\ + \Big\langle\Big\{\lambda u(t) - D\Psi_t^*\big(H_u(t)\big)\Big\}, \Big\{\frac{d\sigma}{dt}(t) + D\Lambda_t\big(u(t)\big)\cdot\big(h(t)\big)\Big\}\Big\rangle_{X\times X^*}\Bigg\}\,dt\,, \tag{3.10}$$

where $\sigma(t) := I\cdot(h(t))$ with $I := \widetilde{T}\circ T : X \to X^*$ and we denote

$$H_u(t) := -\frac{dv}{dt}(t) - \Lambda_t\big(u(t)\big) \in X^* \quad \forall t\in(a,b)\,. \tag{3.11}$$

Proof. First of all by (3.2) it is easy to deduce that

$$\big\|\Lambda_t(x)\big\|_{X^*} \leq g\big(\|T\cdot x\|_H\big)\,\Big(\|x\|_X^{q-1}+1\Big) + \big\|\Lambda_t(0)\big\|_{X^*} \quad \forall x\in X,\ \forall t\in[a,b]\,, \tag{3.12}$$

for some nondecreasing function $g(s) : [0,+\infty) \to (0+\infty)$. Therefore, using (3.6), for every $u\in\mathcal{R}_q(a,b)$ we obtain

$$\Psi_t^*\Big(-\frac{dv}{dt}(t) - \Lambda_t\big(u(t)\big)\Big) \leq L\Big\{\Big\|\frac{dv}{dt}(t)\Big\|_{X^*}^{q^*} + \Big(g\big(\|w(t)\|_H\big)\Big)^{q^*}\Big(\|u(t)\|_X^q + \|\Lambda_t(0)\|_{X^*}^{q^*}\Big)\Big\} \quad \forall t\in[a,b], \tag{3.13}$$

for some constant $L>0$. Since $w(t)\in L^\infty(a,b;H)$, using above inequality and (3.1) gives that $J(u)<+\infty$.

Next clearly for every $u,h\in\mathcal{R}_q(a,b)$ and every $s\in\mathbb{R}$ we have $(u+sh)\in\mathcal{R}_q(a,b)$ i.e. $\mathcal{R}_q(a,b)$ is a linear space. Furthermore, observe that

$$\big\|T\cdot u(t) + s\,T\cdot h(t)\big\|_H \leq M \qquad \forall s\in[-1,1],\ \forall t\in(a,b)\,, \tag{3.14}$$

with $M>0$ independent on t and s. We claim that for a.e. $t\in[a,b]$,

$$D\Psi_t^*\big(H_{(u+sh)}(t)\big) \rightharpoonup D\Psi_t^*\big(H_u(t)\big) \quad \text{weakly in } X \quad \text{as } s\to 0\,. \tag{3.15}$$

Indeed by (2.13) and (2.14) in Lemma 2.3, for some $\bar{C}_0, \bar{C} > 0$ we have

$$\|D\Psi_t(x)\|_{X^*} \le \bar{C}_0 \|x\|^{q-1} + \bar{C}_0 \quad \forall x \in X,\ \forall t \in [a,b]\,, \tag{3.16}$$

and

$$\|D\Psi_t^*(y)\|_{X} \le \bar{C} \|y\|^{q^*-1} + \bar{C} \quad \forall y \in X^*,\ \forall t \in [a,b]\,. \tag{3.17}$$

However, since every bounded sequence of elements of a reflexive Banach space has a subsequence which converges weakly, by (3.13), (3.6) and (3.17), for a.e. fixed t and for every sequence of real numbers $s_n \to 0$, up to a subsequence, we have

$$D\Psi_t^*\big(H_{(u+s_n h)}(t)\big) \rightharpoonup x_0 \quad \text{weakly in } X \quad \text{as } s_n \to 0\,. \tag{3.18}$$

On the other hand, since Ψ_t^* is a convex function, by (2.8) we have

$$\Psi_t^*(y) \ge \Psi_t^*\big(H_{(u+s_n h)}(t)\big) + \Big\langle D\Psi_t^*\big(H_{(u+s_n h)}(t)\big), y - H_{(u+s_n h)}(t)\Big\rangle_{X\times X^*} \quad \forall y \in X^*\,. \tag{3.19}$$

Thus letting $s_n \to 0$ in (3.19) and using (3.18) we deduce,

$$\Psi_t^*(y) \ge \Psi_t^*\big(H_u(t)\big) + \big\langle x_0, y - H_u(t)\big\rangle_{X\times X^*} \quad \forall y \in X^*\,. \tag{3.20}$$

Therefore,

$$\Psi_t^*\big(H_u(t)\big) - \big\langle x_0, H_u(t)\big\rangle_{X\times X^*} = \inf\Big\{\Psi_t^*(y) - \big\langle x_0, y\big\rangle_{X\times X^*} :\ y \in X^*\Big\}\,. \tag{3.21}$$

So $\Psi_t^*\big(H_u(t)\big)$ is a minimizer to the problem in the r.h.s. of (3.21) and thus satisfies the corresponding Euler-Lagrange equation $D\Psi_t^*\big(H_u(t)\big) = x_0$. Therefore, using (3.18), since $s_n \to 0$ was arbitrary sequence we deduce (3.15).

Next clearly for every fixed $t \in [a,b]$ we have

$$\lim_{s\to 0}\frac{1}{s}\Big\{\Psi_t\Big(\lambda\big(u(t)+sh(t)\big)\Big) - \Psi_t\big(\lambda u(t)\big)\Big\} = \Big\langle h(t), \lambda D\Psi_t\big(\lambda u(t)\big)\Big\rangle_{X\times X^*}\,, \tag{3.22}$$

and

$$\lim_{s\to 0}\frac{1}{s}\Big(H_{(u+sh)}(t) - H_u(t)\Big) = -\Big\{\frac{d\sigma}{dt}(t) + D\Lambda_t\big(u(t)\big)\cdot\big(h(t)\big)\Big\}\,, \tag{3.23}$$

where the last limit is taken in the X^*-strong topology. Then in particular

$$\begin{aligned}\lim_{s\to 0}\frac{1}{s}\Big\{\Big\langle u(t)+sh(t), H_{(u+sh)}(t)\Big\rangle_{X\times X^*} - \Big\langle u(t), H_u(t)\Big\rangle_{X\times X^*}\Big\} \\ = \Big\langle h(t), H_u(t)\Big\rangle_{X\times X^*} - \Big\langle u(t), \Big\{\frac{d\sigma}{dt}(t) + D\Lambda_t\big(u(t)\big)\cdot\big(h(t)\big)\Big\}\Big\rangle_{X\times X^*}\,.\end{aligned} \tag{3.24}$$

Next since Ψ_t^* is a convex functions, as before, we have

$$\begin{aligned}\Psi_t^*\big(H_{(u+sh)}(t)\big) - \Psi_t^*\big(H_u(t)\big) \le \Big\langle D\Psi_t^*\big(H_{(u+sh)}(t)\big), H_{(u+sh)}(t) - H_u(t)\Big\rangle_{X\times X^*} \quad \forall y \in X^*\,,\\ \Psi_t^*\big(H_{(u+sh)}(t)\big) - \Psi_t^*\big(H_u(t)\big) \ge \Big\langle D\Psi_t^*\big(H_u(t)\big), H_{(u+sh)}(t) - H_u(t)\Big\rangle_{X\times X^*} \quad \forall y \in X^*\,.\end{aligned} \tag{3.25}$$

Therefore, by (3.25), (3.23) and (3.15) we deduce

$$\lim_{s\to 0}\frac{1}{s}\Big\{\Psi_t^*\big(H_{(u+sh)}(t)\big) - \Psi_t^*\big(H_u(t)\big)\Big\} = -\Big\langle D\Psi_t^*\big(H_u(t)\big), \Big\{\frac{d\sigma}{dt}(t) + D\Lambda_t\big(u(t)\big)\cdot\big(h(t)\big)\Big\}\Big\rangle_{X\times X^*}\,. \tag{3.26}$$

On the other hand, using (3.16), by (3.22) and the Lagrange Theorem from the elementary Calculus we deduce that for every $t \in [a,b]$ and every $s \in [-1,1]$, there exists τ in the interval with the endpoints 0 and s, such that

$$\left|\frac{1}{s}\left\{\Psi_t\Big(\lambda\big(u(t)+sh(t)\big)\Big)-\Psi_t\big(\lambda u(t)\big)\right\}\right| \leq \\ \|h(t)\|_X\cdot\left\|\lambda D\Psi_t\Big(\lambda\big(u(t)+\tau h(t)\big)\Big)\right\|_{X^*} \leq C\Big(\|h(t)\|_X^q+\|u(t)\|_X^q+1\Big)\,, \tag{3.27}$$

for some constant $C>0$ independent on t and s. Similarly, using (3.2), (3.12) and (3.14), from (3.24) we deduce

$$\left|\frac{1}{s}\left\{\Big\langle u(t)+sh(t),H_{(u+sh)}(t)\Big\rangle_{X\times X^*}-\Big\langle u(t),H_u(t)\Big\rangle_{X\times X^*}\right\}\right| \leq \\ \|h(t)\|_X\cdot\big\|H_{u+\tau h}(t)\big\|_{X^*}+\big\|u(t)+\tau h(t)\big\|_X\cdot\left\|\frac{d\sigma}{dt}(t)+D\Lambda_t\Big(u(t)+\tau h(t)\Big)\cdot\big(h(t)\big)\right\|_{X^*} \\ \leq \bar{C}\Big(\|u(t)\|_X+\|h(t)\|_X\Big)\cdot\left(\|u(t)\|_X^{q-1}+\|h(t)\|_X^{q-1}+\left\|\frac{dv}{dt}(t)\right\|_{X^*}+\left\|\frac{d\sigma}{dt}(t)\right\|_{X^*}+\|\Lambda_t(0)\|_{X^*}+1\right) \\ \leq C_0\left(\|u(t)\|_X^q+\|h(t)\|_X^q+\left\|\frac{dv}{dt}(t)\right\|_{X^*}^{q^*}+\left\|\frac{d\sigma}{dt}(t)\right\|_{X^*}^{q^*}+\|\Lambda_t(0)\|_{X^*}^{q^*}+1\right)\,, \tag{3.28}$$

where the constants $\bar{C}, C_0>0$ are independent on t and s. Finally by the same way, using (3.17), (3.13) and the fact that $\Lambda_t(0)\in L^{q^*}(a,b;X^*)$, from (3.26) we infer

$$\left|\frac{1}{s}\left\{\Psi_t^*\big(H_{(u+sh)}(t)\big)-\Psi_t^*\big(H_u(t)\big)\right\}\right| \leq \big\|D\Psi_t^*\big(H_{u+\tau h}(t)\big)\big\|_X\cdot\left\|\frac{d\sigma}{dt}(t)+D\Lambda_t\Big(u(t)+\tau h(t)\Big)\cdot\big(h(t)\big)\right\|_{X^*} \leq \\ C\left(\|u(t)\|_X^{q-1}+\|h(t)\|_X^{q-1}+\left\|\frac{dv}{dt}(t)\right\|_{X^*}+\left\|\frac{d\sigma}{dt}(t)\right\|_{X^*}+\|\Lambda_t(0)\|_X+1\right)^{q^*-1}\times \\ \left(\|u(t)\|_X^{q-1}+\|h(t)\|_X^{q-1}+\left\|\frac{d\sigma}{dt}(t)\right\|_{X^*}+1\right) \leq C_1\left(\|u(t)\|_X^q+\|h(t)\|_X^q+\left\|\frac{dv}{dt}(t)\right\|_{X^*}^{q^*}+\left\|\frac{d\sigma}{dt}(t)\right\|_{X^*}^{q^*}+\tilde{\mu}(t)\right), \tag{3.29}$$

where again the constant $C_1>0$ is independent on t and s and $\tilde{\mu}(t)\in L^1(a,b;\mathbb{R})$. However, by the definition of $\mathcal{R}_q(a,b)$ we have $u,h\in L^q(a,b;X)$ and $\frac{dv}{dt},\frac{d\sigma}{dt}\in L^{q^*}(a,b;X^*)$. Therefore, using the Dominated Convergence Theorem, by (3.7), (3.22) and (3.27), (3.24) and (3.28), (3.26) and (3.29), we deduce (3.10). □

Theorem 3.1. *Let $\{X,H,X^*\}$ be an evolution triple with the corresponding inclusion operator $T\in\mathcal{L}(X;H)$, as it was defined in Definition 2.9, together with the corresponding operator $\widetilde{T}\in\mathcal{L}(H;X^*)$, defined as in* (2.62)*. Furthermore, let $a,b,q,\lambda\in\mathbb{R}$ be such that $a<b$, $q\geq 2$ and $\lambda\in\{0,1\}$. Assume that Ψ_t and Λ_t satisfy all the conditions of Definition 3.2. Furthermore let $w_0\in H$ and $\mathcal{R}_q(a,b)$ and J be as in Definitions 3.1 and 3.2 respectively. Consider the minimization problem*

$$\inf\{J(u):\,u\in\mathcal{R}_q(a,b),\,w(a)=w_0\}\,, \tag{3.30}$$

where $w(t):=T\cdot\big(u(t)\big)$. Then for $u\in\mathcal{R}_q(a,b)$ such that $w(a)=w_0$ the following four statements are equivalent:

- **(a)** *u is a minimizer to* (3.30)*.*
- **(b)** *u is a critical point of* (3.30) *i.e. for an arbitrary function $h(t)\in\mathcal{R}_q(a,b)$, such that $T\cdot h(a)=0$ we have*

$$\lim_{s\to 0}\frac{J(u+sh)-J(u)}{s}=0\,. \tag{3.31}$$

(c) *u is a solution to*

$$\frac{dv}{dt}(t) + \Lambda_t\big(u(t)\big) + D\Psi_t\big(\lambda u(t)\big) = 0 \quad \text{for a.e. } t \in (a,b)\,, \tag{3.32}$$

where $v(t) := I \cdot \big(u(t)\big) = \widetilde{T} \cdot \big(w(t)\big)$ with $I := \widetilde{T} \circ T : X \to X^*$.

(d) $J(u) = 0$.

Proof. Assume that u is a minimizer to (3.30). Let $h(t) \in \mathcal{R}_q(a,b)$ be an arbitrary function, such that $T \cdot h(a) = 0$. Here we assume that $T \cdot (h(t))$ is H-weakly continuous on $[a,b]$. Then for every $s \in \mathbb{R}$ $\{u(t) + sh(t)\} \in \mathcal{R}_q(a,b)$ and $T \cdot \big(u(a) + sh(a)\big) = w_0$. Therefore, since u is a minimizer we deduce that $f_h(s) := J(u + sh) \geq J(u) = f_h(0)$ for every $s \in \mathbb{R}$. Thus by the elementary Calculus we must have $f_h'(0) = 0$ (remember that, by Lemma 3.1, f_h is diferrentiable function). So we obtain (3.31).

Next assume that for some $u \in \mathcal{R}_q(a,b)$ such that $w(a) = w_0$ we have (3.31) i.e.

$$\lim_{s\to 0} \frac{J(u+sh) - J(u)}{s} = 0\,. \tag{3.33}$$

for every $h(t) \in \mathcal{R}_q(a,b)$, such that $T \cdot h(a) = 0$. Then, by (3.10) in Lemma 3.1, for every $h(t) \in \mathcal{R}_q(a,b)$, such that $T \cdot h(a) = 0$ we must have

$$\int_a^b \Bigg\{ \Big\langle h(t), \lambda\Big\{D\Psi_t\big(\lambda u(t)\big) - H_u(t)\Big\}\Big\rangle_{X\times X^*} + \Big\langle \Big\{\lambda u(t) - D\Psi_t^*\big(H_u(t)\big)\Big\}, \Big\{\frac{d\sigma}{dt}(t) + D\Lambda_t\big(u(t)\big) \cdot \big(h(t)\big)\Big\}\Big\rangle_{X\times X^*} \Bigg\} dt = 0\,, \tag{3.34}$$

where $\sigma(t) := I \cdot (h(t))$ with $I := \widetilde{T} \circ T : X \to X^*$ and we, as before, denote

$$H_u(t) := -\frac{dv}{dt}(t) - \Lambda_t\big(u(t)\big) \in X^* \quad \forall t \in (a,b)\,. \tag{3.35}$$

Next as in (3.16) we have

$$\|D\Psi_t(x)\|_{X^*} \leq \bar{C}_0 \|x\|^{q-1} + \bar{C}_0 \quad \forall x \in X,\ \forall t \in [a,b]\,, \tag{3.36}$$

and as in (3.17) we have

$$\|D\Psi_t^*(y)\|_X \leq \bar{C} \|y\|^{q^*-1} + \bar{C} \quad \forall y \in X^*,\ \forall t \in [a,b]\,, \tag{3.37}$$

and since we have $\int_a^b \Psi_t^*\big(H_u(t)\big)dt < +\infty$, by (3.6) we obtain $H_u(t) \in L^{q^*}(a,b;X^*)$. Therefore, if we set

$$W_u(t) := \lambda u(t) - D\Psi_t^*\big(H_u(t)\big) \qquad \forall t \in [a,b]\,, \tag{3.38}$$

then, since $u(t) \in L^q(a,b;X)$, we deduce

$$W_u(t) \in L^q(a,b;X)\,. \tag{3.39}$$

On the other hand, by Remark 3.1, $x \in X$ satisfies $x = D\Psi_t^*(y)$ if and only if $y \in X^*$ satisfies $y = D\Psi_t(x)$. Therefore, by (3.34) we have

$$\int_a^b \Bigg(\Big\langle \lambda h(t), \Big\{ D\Psi_t\Big(D\Psi_t^*\big(H_u(t)\big) + W_u(t)\Big) - D\Psi_t\Big(D\Psi_t^*\big(H_u(t)\big)\Big)\Big\}\Big\rangle_{X\times X^*} + \Big\langle W_u(t), D\Lambda_t\big(u(t)\big) \cdot \big(h(t)\big)\Big\rangle_{X\times X^*} + \Big\langle W_u(t), \frac{d\sigma}{dt}(t)\Big\rangle_{X\times X^*} \Bigg) dt = 0\,. \tag{3.40}$$

Next for every $t \in [a,b]$ let $\big(D\Lambda_t\big(u(t)\big)\big)^* \in \mathcal{L}(X,X^*)$ be the adjoint to $D\Lambda_t\big(u(t)\big) \in \mathcal{L}(X,X^*)$ operator defined by

$$\Big\langle x_2, \big(D\Lambda_t\big(u(t)\big)\big)^* \cdot x_1\Big\rangle_{X\times X^*} := \Big\langle x_1, D\Lambda_t\big(u(t)\big) \cdot x_2\Big\rangle_{X\times X^*} \quad \forall\, x_1, x_2 \in X\,. \tag{3.41}$$

Define the strictly measurable function $P_u(t) : (a,b) \to X^*$ by

$$P_u(t) := \lambda\Big\{D\Psi_t\Big(D\Psi_t^*\big(H_u(t)\big) + W_u(t)\Big) - D\Psi_t\Big(D\Psi_t^*\big(H_u(t)\big)\Big)\Big\} + \Big(D\Lambda_t\big(u(t)\big)\Big)^* \cdot \big(W_u(t)\big)\,. \tag{3.42}$$

Then, since by (3.41) we have

$$\begin{aligned}\Big\langle x, P_u(t)\Big\rangle_{X\times X^*} = \Big\langle \lambda x, \Big\{D\Psi_t\Big(D\Psi_t^*\big(H_u(t)\big) + W_u(t)\Big) - D\Psi_t\Big(D\Psi_t^*\big(H_u(t)\big)\Big)\Big\}\Big\rangle_{X\times X^*} \\ + \Big\langle W_u(t), D\Lambda_t\big(u(t)\big) \cdot x\Big\rangle_{X\times X^*} \qquad \forall x \in X\,,\end{aligned} \tag{3.43}$$

we deduce from (3.40),

$$\int_a^b \Big\langle h(t), P_u(t)\Big\rangle_{X\times X^*} dt + \int_a^b \Big\langle W_u(t), \frac{d\sigma}{dt}(t)\Big\rangle_{X\times X^*} dt = 0\,. \tag{3.44}$$

On the other hand, using the fact $T\cdot u(t) \in L^\infty(a,b;H)$, by (3.43), (3.36), (3.37) and (3.2) we deduce

$$\big\|P_u(t)\big\|_{X^*} \leq C\Big\{\big\|H_u(t)\big\|_{X^*} + \big\|W_u(t)\big\|_X^{q-1} + \big\|u(t)\big\|_X^{q-1} + 1\Big\}\,,$$

where the constant $C > 0$ doesn't depend on t. Thus since $H_u(t) \in L^{q^*}(a,b;X^*)$, $u(t) \in L^q(a,b;X)$ and $W_u(t) \in L^q(a,b;X)$ we infer

$$P_u(t) \in L^{q^*}(a,b;X^*)\,. \tag{3.45}$$

Next remember that we established (3.44) for every $h(t) \in \mathcal{R}_q(a,b)$, such that $T \cdot h(a) = 0$. In particular (3.44) holds for $h(t) = \delta(t)$ where $\delta(t) \in C^1\big((a,b);X\big)$ satisfying $\operatorname{supp}\delta \subset\subset (a,b)$. For such $\delta(t)$ we have

$$\int_a^b \Big\langle \delta(t), P_u(t)\Big\rangle_{X\times X^*} dt = -\int_a^b \Big\langle W_u(t), I \cdot \Big(\frac{d\delta}{dt}(t)\Big)\Big\rangle_{X\times X^*} dt = -\int_a^b \Big\langle \frac{d\delta}{dt}(t), I \cdot \big(W_u(t)\big)\Big\rangle_{X\times X^*} dt\,. \tag{3.46}$$

Thus, by (3.46), (3.45) and by Definition 2.4, $I \cdot \big(W_u(t)\big)$ belongs to $W^{1,q^*}(a,b;X^*)$ and

$$\frac{d(I \cdot W_u)}{dt}(t) = P_u(t)\,. \tag{3.47}$$

Thus since $W_u(t) \in L^q(a,b;X)$ and $P_u(t) \in L^{q^*}(a,b;X^*)$ we have $W_u \in \mathcal{R}_q(a,b)$. Then by Lemma 2.8, the function $L_u(t) : [a,b] \to H$ defined by $L_u(t) := T \cdot \big(W_u(t)\big)$ belongs to $L^\infty(a,b;H)$ and, up to a redefinition of $L_u(t)$ on a subset of $[a,b]$ of Lebesgue measure zero, L_u is H-weakly continuous, as it was stated in Corollary 2.1. Moreover, by the same Corollary for every $a \leq \alpha < \beta \leq b$ and for every $\delta(t) \in C^1\big([a,b];X\big)$ we will have

$$\int_\alpha^\beta \Big\{\Big\langle \delta(t), P_u(t)\Big\rangle_{X\times X^*} + \Big\langle \frac{d\delta}{dt}(t), I\cdot\big(W_u(t)\big)\Big\rangle_{X\times X^*}\Big\}dt = \big\langle T\cdot\delta(\beta), L_u(\beta)\big\rangle_{H\times H} - \big\langle T\cdot\delta(\alpha), L_u(\alpha)\big\rangle_{H\times H}\,. \tag{3.48}$$

Thus in particular for every $h_0(t) \in C^1\big([a,b];X\big)$ such that $h_0(a)=0$ we have

$$\int\limits_a^b \Big\langle h_0(t), P_u(t)\Big\rangle_{X\times X^*} dt = -\int\limits_a^b \Big\langle \frac{dh_0}{dt}(t), I\cdot\big(W_u(t)\big)\Big\rangle_{X\times X^*} dt + \big\langle T\cdot h_0(b), L_u(b)\big\rangle_{H\times H}. \tag{3.49}$$

However, inserting $h := h_0$ into (3.44) gives

$$\int\limits_a^b \Big\langle h_0(t), P_u(t)\Big\rangle_{X\times X^*} dt = -\int\limits_a^b \Big\langle W_u(t), \frac{d(I\cdot h_0)}{dt}(t)\Big\rangle_{X\times X^*} dt = -\int\limits_a^b \Big\langle \frac{dh_0}{dt}(t), I\cdot\big(W_u(t)\big)\Big\rangle_{X\times X^*} dt. \tag{3.50}$$

Comparing (3.49) with (3.50) we obtain

$$\big\langle T\cdot h_0(b), L_u(b)\big\rangle_{H\times H} = 0. \tag{3.51}$$

In particular we can test (3.51) with $h_0(t) := (t-a)x$ for arbitrary $x \in X$. Then we obtain

$$\big\langle Tx, L_u(b)\big\rangle_{H\times H} = 0 \quad \forall x \in X, \tag{3.52}$$

and since T has dense image in H we finally get

$$L_u(b) = 0. \tag{3.53}$$

Thus, since $L_u(t) := T\cdot\big(W_u(t)\big)$, using (3.47), (3.53) and the fact that $W_u \in \mathcal{R}_q(a,b)$, by Lemma 2.8 we deduce

$$\int_\alpha^b \Big\langle W_u(t), P_u(t)\Big\rangle_{X\times X^*} dt = -\frac{1}{2}\|L_u(\alpha)\|_H^2 \quad \forall \alpha \in [a,b]. \tag{3.54}$$

Then plugging (3.43) into (3.54), by (3.38) we infer

$$\begin{aligned} &-\frac{1}{2}\Big\|T\cdot\big(W_u(\alpha)\big)\Big\|_H^2 = \\ &\int\limits_\alpha^b \left(\Big\langle W_u(t), \lambda\Big\{D\Psi_t\Big(D\Psi_t^*\big(H_u(t)\big)+W_u(t)\Big)-D\Psi_t\Big(D\Psi_t^*\big(H_u(t)\big)\Big)\Big\}+D\Lambda_t\big(u(t)\big)\cdot\big(W_u(t)\big)\Big\rangle_{X\times X^*}\right)dt \\ &= \int\limits_\alpha^b \left(\Big\langle W_u(t), \lambda\Big\{D\Psi_t\big(\lambda u(t)\big)-D\Psi_t\Big(\lambda u(t)-W_u(t)\Big)\Big\}+D\Lambda_t\big(u(t)\big)\cdot\big(W_u(t)\big)\Big\rangle_{X\times X^*}\right)dt \quad \forall \alpha \in [a,b]. \end{aligned} \tag{3.55}$$

Next, by the condition (3.3) in the Definition 3.2 we have

$$\Big\langle h, \lambda\Big\{D\Psi_t\big(\lambda x\big) - D\Psi_t(\lambda x - h)\Big\} + D\Lambda_t(x)\cdot h\Big\rangle_{X\times X^*} \geq -\hat{g}\big(\|T\cdot x\|_H\big)\Big(\|x\|_X^q + \mu(t)\Big)\,\|T\cdot h\|_H^2 \quad \forall x,h \in X,\ \forall t\in[a,b], \tag{3.56}$$

for some non-decreasing function $\hat{g}(s) : [0+\infty) \to (0,+\infty)$ and some nonnegative function $\mu(t) \in L^1(a,b;\mathbb{R})$. Therefore, using (3.55), we deduce

$$\frac{1}{2}\Big\|T\cdot\big(W_u(\alpha)\big)\Big\|_H^2 \leq \int_\alpha^b \hat{g}\Big(\big\|T\cdot\big(u(t)\big)\big\|_H\Big)\Big(\big\|u(t)\big\|_X^q + \mu(t)\Big)\Big\|T\cdot\big(W_u(t)\big)\Big\|_H^2 dt \quad \forall \alpha \in [a,b]. \tag{3.57}$$

In particular, since $T\cdot u \in L^\infty(a,b;H)$ we clearly obtain

$$\frac{1}{2}\Big\|T\cdot\big(W_u(t)\big)\Big\|_H^2 \leq K\int_t^b \Big(\big\|u(s)\big\|_X^q + \mu(s)\Big)\Big\|T\cdot\big(W_u(s)\big)\Big\|_H^2 ds \quad \forall t \in [a,b], \tag{3.58}$$

for some $K > 0$ independent on t. Next define $g(t) := \Big(\big\|u(t)\big\|_X^q + \mu(t)\Big)$ and $h(t) := g(t)\Big\|T\cdot\big(W_u(t)\big)\Big\|_H^2$. Then since $u(t) \in L^q(a,b;X)$ and $T\cdot\big(W_u(t)\big) \in L^\infty(a,b;H)$ we clearly have $g(t), h(t) \in L^1(a,b;\mathbb{R})$. Moreover, by (3.58), we infer

$$h(t) \leq 2Kg(t)\int_t^b h(s)\,ds \qquad \text{for a.e. } t \in [a,b]\,. \tag{3.59}$$

On the other hand, the function $\gamma(t) := e^{-2K\int_t^b g(s)ds}\cdot\int_t^b h(s)\,ds \in W^{1,1}(a,b;\mathbb{R})$ and by (3.59), we obtain

$$\frac{d}{dt}\bigg(e^{-2K\int_t^b g(s)ds}\cdot\int_t^b h(s)\,ds\bigg) \geq 0 \qquad \text{for a.e. } t \in [a,b]\,.$$

Therefore,

$$e^{-2K\int_a^b g(s)ds}\cdot\int_a^b h(s)\,ds \leq e^{-2K\int_b^b g(s)ds}\cdot\int_b^b h(s)\,ds = 0\,.$$

Therefore, since $h(t) \geq 0$ we obtain $h(t) = 0$ for a.e. $t \in [a,b]$. Thus $\big\|T\cdot\big(W_u(t)\big)\big\|_H^2 = 0$ for a.e. $t \in [a,b]$. Therefore, since T is injective, by the definition of W_u in (3.38) we have $\lambda u(t) = D\Psi_t^*\big(H_u(t)\big)$ for a.e. $t \in [a,b]$. I.e.

$$D\Psi_t\big(\lambda u(t)\big) = H_u(t) = -\frac{dv}{dt}(t) - \Lambda_t\big(u(t)\big) \qquad \text{for a.e. } t \in [a,b]\,, \tag{3.60}$$

So u is a solution to (3.32).

Finally if u is a solution to (3.32) then by Remark 3.1 we have $J(u) = 0$. Moreover, by Remark 3.1 we always have $J(\cdot) \geq 0$. Thus if we have $J(u) = 0$ then trivially u is a minimizer to (3.30). □

The following proposition provides uniqueness of the solution.

Proposition 3.1. *Let $\{X, H, X^*\}$ be an evolution triple with the corresponding inclusion operator $T \in \mathcal{L}(X;H)$, as it was defined in Definition 2.9, together with the corresponding operator $\widetilde{T} \in \mathcal{L}(H;X^*)$, defined as in (2.62). Furthermore, let $a, b, q \in \mathbb{R}$ be such that $a < b$, $q \geq 2$ and $\lambda \in \{0,1\}$. Assume that Ψ_t and Λ_t satisfy all the conditions of Definition 3.2. Then for every $w_0 \in H$ there exists at most one function $u(t) \in L^q(a,b;X)$, such that $w(t) := T\cdot\big(u(t)\big) \in L^\infty(a,b;H)$, $v(t) := \widetilde{T}\cdot\big(w(t)\big) = \widetilde{T}\circ T\big(u(t)\big) \in W^{1,q^*}(a,b;X^*)$,where $q^* := q/(q-1)$, and $u(t)$ is a solution to*

$$\begin{cases} \frac{dv}{dt}(t) + \Lambda_t\big(u(t)\big) + D\Psi_t\big(\lambda u(t)\big) = 0 & \textit{for a.e. } t \in (a,b)\,,\\ w(a) = w_0\,. \end{cases} \tag{3.61}$$

Proof. Let $w_0 \in H$ and let $\hat{u}(t), \tilde{u}(t) \in L^q(a,b;X)$ be such that $\hat{w}(t) := T\cdot\big(\hat{u}(t)\big) \in L^\infty(a,b;H)$, $\tilde{w}(t) := T\cdot\big(\tilde{u}(t)\big) \in L^\infty(a,b;H)$, $\hat{v}(t) := \widetilde{T}\cdot\big(\hat{w}(t)\big) = \widetilde{T}\circ T\big(\hat{u}(t)\big) \in W^{1,q^*}(a,b;X^*)$, $\tilde{v}(t) := \widetilde{T}\cdot\big(\tilde{w}(t)\big) = \widetilde{T}\circ T\big(\tilde{u}(t)\big) \in W^{1,q^*}(a,b;X^*)$, where $q^* := q/(q-1)$, and $\hat{u}(t), \tilde{u}(t)$ are both solutions to (3.61). Then the function $\underline{u}(t) := \big(\hat{u}(t) - \tilde{u}(t)\big) \in L^q(a,b;X)$ is such that $\underline{w}(t) := T\cdot\big(\underline{u}(t)\big) \in L^\infty(a,b;H)$, $\underline{v}(t) := \widetilde{T}\cdot\big(\underline{w}(t)\big) = \widetilde{T}\circ T\big(\underline{u}(t)\big) \in W^{1,q^*}(a,b;X^*)$, $\underline{w}(a) = 0$ and by (3.61), and the facts that $\lambda \in \{0,1\}$ and $\Psi_t(0) = 0$, we obtain

$$\frac{d\underline{v}}{dt}(t) + \Big\{\Lambda_t\big(\hat{u}(t)\big) - \Lambda_t\big(\tilde{u}(t)\big)\Big\} + \Big\{D\Psi_t\big(\lambda\hat{u}(t)\big) - D\Psi_t\big(\lambda\tilde{u}(t)\big)\Big\} = 0 \qquad \text{for a.e. } t \in (a,b)\,. \tag{3.62}$$

Thus plugging (3.3) into (3.62) we deduce

$$\int_a^\tau \bigg\{\Big\langle\underline{u}(t), \frac{d\underline{v}}{dt}(t)\Big\rangle_{X\times X^*} - \bar{\gamma}(t)\|\underline{w}(t)\|_H^2\bigg\}dt \leq 0 \quad \text{for every } \tau \in [a,b]\,, \tag{3.63}$$

where $\bar{\gamma}(t) \in L^1\big(a,b;[0,+\infty)\big)$. On the other hand, since $\underline{w}(a)=0$, by Lemma 2.8 we have

$$\int_a^\tau \Big\langle \underline{u}(t), \frac{d\underline{v}}{dt}(t)\Big\rangle_{X\times X^*} dt = \frac{1}{2}\big\|\underline{w}(\tau)\big\|_H^2 \quad \text{for every } \tau\in[a,b]\,.$$

Thus, inserting it into (3.63), we deduce

$$\big\|\underline{w}(\tau)\big\|_H^2 \le C\int_a^\tau \bar{\gamma}(t)\big\|\underline{w}(t)\big\|_H^2 dt \quad \text{for every } \tau\in[a,b]\,. \tag{3.64}$$

Therefore, exactly as before in the end of the proof of Theorem 3.1, by (3.64) we deduce $\underline{w}(t)=0$ for a.e. $t\in[a,b]$ and since T is an injective operator we have $\hat{u}(t)=\tilde{u}(t)$ for a.e. $t\in[a,b]$. This completes the proof. □

Definition 3.3. Let $\{X,H,X^*\}$ be an evolution triple with the corresponding inclusion operator $T\in\mathcal{L}(X;H)$, as it was defined in Definition 2.9, together with the corresponding operator $\widetilde{T}\in\mathcal{L}(H;X^*)$, defined as in (2.62). Furthermore, let $a,b,q\in\mathbb{R}$ be s.t. $a<b$ and $q\ge 2$. Next assume that Ψ_t and Λ_t satisfy all the conditions of Definition 3.2 together with the assumption $\lambda=1$. Moreover, assume that Ψ_t and Λ_t satisfy the following positivity condition

$$\Psi_t(x)+\Big\langle x,\Lambda_t(x)\Big\rangle_{X\times X^*} \ge \frac{1}{\bar{C}}\,\|x\|_X^q - \bar{C}\Big(\|x\|_X^r+1\Big)\Big(\|T\cdot x\|_H^{(2-r)}+1\Big)-\bar{\mu}(t) \quad \forall x\in X,\ \forall t\in[a,b], \tag{3.65}$$

where $r\in[0,2)$ and $\bar{C}>0$ are some constants and $\bar{\mu}(t)\in L^1(a,b;\mathbb{R})$ is some nonnegative function. Furthermore, assume that

$$\Lambda_t(x) = A_t\big(S\cdot x\big)+\Theta_t(x) \qquad \forall\, x\in X,\ \forall\, t\in[a,b], \tag{3.66}$$

where Z is a Banach space, $S:X\to Z$ is a compact operator, for every $t\in[a,b]$ $A_t(z):Z\to X^*$ and $\Theta_t(x):X\to X^*$ are functions, such that A_t is strongly Borel on the pair of variables (z,t) and Gateaux differentiable at every $z\in Z$, Θ_t is strongly Borel on the pair of variables (x,t) and Gateaux differentiable at every $x\in X$, $\Theta_t(0), A_t(0)\in L^{q^*}\big((a,b);X^*\big)$ and the derivatives of A_t and Θ_t satisfy the growth conditions

$$\|DA_t(S\cdot x)\|_{\mathcal{L}(Z;X^*)} \le g\big(\|T\cdot x\|\big)\left(\|x\|_X^{q-2}+1\right) \quad \forall x\in X,\ \forall t\in[a,b]\,, \tag{3.67}$$

$$\|D\Theta_t(x)\|_{\mathcal{L}(X;X^*)} \le g\big(\|T\cdot x\|\big)\left(\|x\|_X^{q-2}+1\right) \quad \forall x\in X,\ \forall t\in[a,b]\,, \tag{3.68}$$

for some nondecreasing function $g(s):[0,+\infty)\to(0+\infty)$. Finally assume that for every sequence $\big\{x_n(t)\big\}_{n=1}^{+\infty}\subset L^q(a,b;X)$ such that the sequence $\big\{(\widetilde{T}\circ T)\cdot x_n(t)\big\}$ is bounded in $W^{1,q^*}(a,b;X^*)$ and $x_n(t)\rightharpoonup x(t)$ weakly in $L^q(a,b;X)$ we have

- $\Theta_t\big(x_n(t)\big)\rightharpoonup\Theta_t\big(x(t)\big)$ weakly in $L^{q^*}(a,b;X^*)$,
- $\underline{\lim}_{n\to+\infty}\int_a^b \Big\langle x_n(t),\Theta_t\big(x_n(t)\big)\Big\rangle_{X\times X^*} dt \ge \int_a^b \Big\langle x(t),\Theta_t\big(x(t)\big)\Big\rangle_{X\times X^*} dt.$

Next, as in (3.4) with $\lambda=1$, let $J_0(u):\mathcal{R}_q(a,b)\to\mathbb{R}$ (where $\mathcal{R}_q(a,b)$ was defined in Definition 3.1) be defined by

$$J_0(u) := \frac{1}{2}\Big(\|w(b)\|_H^2-\|w(a)\|_H^2\Big)+ \int_a^b \left\{\Psi_t\big(u(t)\big)+\Psi_t^*\Big(-\frac{dv}{dt}(t)-\Lambda_t\big(u(t)\big)\Big)+\Big\langle u(t),\Lambda_t\big(u(t)\big)\Big\rangle_{X\times X^*}\right\}dt\,, \tag{3.69}$$

where $w(t):=T\cdot\big(u(t)\big)$, $v(t):=I\cdot\big(u(t)\big)=\widetilde{T}\cdot\big(w(t)\big)$ with $I:=\widetilde{T}\circ T:\ X\to X^*$ and we assume that $w(t)$ is H-weakly continuous on $[a,b]$, as it was stated in Corollary 2.1. Moreover, for every $w_0\in H$ consider the minimization problem

$$\inf\Big\{J_0(\psi):\ u\in\mathcal{R}_q(a,b),\ w(a)=w_0\Big\}\,. \tag{3.70}$$

Remark 3.3. As before, we can rewrite the definition of J_0 in (3.69) by

$$J_0(u) := \int\limits_a^b \left\{\Psi_t\big(u(t)\big) + \Psi_t^*\bigg(-\frac{dv}{dt}(t) - \Lambda_t\big(u(t)\big)\bigg) + \bigg\langle u(t), \frac{dv}{dt}(t) + \Lambda_t\big(u(t)\big)\bigg\rangle_{X\times X^*}\right\}dt\,. \quad (3.71)$$

Proposition 3.2. *Let $J_0(u)$ be as in Definition 3.3 and all the conditions of Definitions 3.3 are satisfied. Moreover let $w_0 \in H$ be such that $w_0 = T\cdot u_0$ for some $u_0 \in X$, or more generally, $w_0 \in H$ be such that $\mathcal{A}_{w_0} := \big\{u \in \mathcal{R}_q(a,b) : w(a) = w_0\big\} \neq \emptyset$. Then there exists a minimizer to (3.70). In particular there exists a unique solution to*

$$\begin{cases} \frac{dv}{dt}(t) + \Lambda_t\big(u(t)\big) + D\Psi_t\big(u(t)\big) = 0 \quad \text{for a.e. } t \in (a,b)\,,\\ w(a) = w_0\,, \end{cases} \quad (3.72)$$

where $w(t) := T\cdot\big(u(t)\big)$, $v(t) := I\cdot\big(u(t)\big) = \widetilde{T}\cdot\big(w(t)\big)$ with $I := \widetilde{T}\circ T : X \to X^*$ and we assume that $w(t)$ is H-weakly continuous on $[a,b]$, as it was stated in Corollary 2.1.

Proof. First of all we would like to note that in the case, $w_0 = T\cdot u_0$ for some $u_0 \in X$, the set

$$\mathcal{A}_{w_0} := \big\{u \in \mathcal{R}_q(a,b) : w(a) = w_0\big\} = \big\{u \in \mathcal{R}_q(a,b) : T\cdot u(a) = T\cdot u_0\big\}$$

is not empty. In particular the function $u_0(t) \equiv u_0$ belongs to $\mathcal{A}_{w_0}$. Thus, in any case $\mathcal{A}_{w_0} \neq \emptyset$. Next let

$$K := \inf_{\theta\in\mathcal{A}_{w_0}} J_0(\theta)\,. \quad (3.73)$$

Then $K \geq 0$. Consider a minimizing sequence $\{u_n(t)\} \subset \mathcal{A}_{w_0}$, i.e. a sequence such that

$$\lim_{n\to\infty} J_0(u_n) = K\,. \quad (3.74)$$

Set $\Upsilon_n(t) : (a,b) \to \mathbb{R}$ by

$$\Upsilon_n(t) := \Psi_t\big(u_n(t)\big) + \Psi_t^*\bigg(-\frac{dv_n}{dt}(t) - \Lambda_t\big(u_n(t)\big)\bigg) + \bigg\langle u_n(t), \frac{dv_n}{dt}(t) + \Lambda_t\big(u_n(t)\big)\bigg\rangle_{X\times X^*}, \quad (3.75)$$

where $w_n(t) := T\cdot\big(u_n(t)\big)$ and $v_n(t) := \widetilde{T}\cdot\big(w_n(t)\big)$. Then by the definition of Legendre transform we deduce that $\Upsilon_n(t) \geq 0$ for a.e. $t \in (a,b)$. On the other hand, by (3.71) we obtain

$$\int_a^b \Upsilon_n(t)dt = J_0(u_n) \to K \qquad \text{as } n \to +\infty\,. \quad (3.76)$$

Therefore, by (3.76) and the fact that $\Upsilon_n(t) \geq 0$ we deduce that there exists a constant $C_0 > 0$ such that for every $n \in \mathbb{N}$ and $t \in [a,b]$ we have

$$\int_a^t \Upsilon_n(s)ds \leq C_0\,. \quad (3.77)$$

However, since by Lemma 2.8 we have

$$\int_a^t \bigg\langle u_n(s), \frac{dv_n}{dt}(s)\bigg\rangle_{X\times X^*} ds = \frac{1}{2}\Big(\|w_n(t)\|_H^2 - \|w_0\|_H^2\Big)\,,$$

plugging (3.75) into (3.77) and using (3.65) gives for every $n \in \mathbb{N}$,

$$\int\limits_a^t \left\{\frac{1}{\hat{C}}\,\|u_n(s)\|_X^q + \Psi_s^*\bigg(-\frac{dv_n}{dt}(s) - \Lambda_s\big(u_n(s)\big)\bigg)\right\}ds + \frac{1}{2}\|w_n(t)\|_H^2$$
$$\leq \frac{1}{2}\|w_0\|_H^2 + \int_a^t \bar{\mu}(s)ds + \hat{C}\int_a^t \Big(\|u_n(s)\|_X^p + 1\Big)\cdot\Big(\|w_n(s)\|_H^{(2-p)} + 1\Big)ds \qquad \forall t \in [a,b]\,, \quad (3.78)$$

where $p \in [0,2)$ and $\hat{C}$ is some positive constant. Therefore, in particular, for every n we have

$$\int_a^t \|u_n(s)\|_X^q ds + \frac{1}{2}\|w_n(t)\|_H^2 \ \le\ \tilde{C}\int_a^t \|u_n(s)\|_X^p \cdot \|w_n(s)\|_H^{(2-p)} ds + \tilde{C} \qquad \forall t \ \in\ [a,b]\,. \quad (3.79)$$

Thus by Hölder inequality, for every n we deduce

$$\int_a^t \|u_n(s)\|_X^q ds+\frac{1}{2}\|w_n(t)\|_H^2 \le \tilde{C}\bigg(\int_a^t \|u_n(s)\|_X^2 dt\bigg)^{\frac{p}{2}}\cdot\bigg(\int_a^t \|w_n(s)\|_H^2 ds\bigg)^{\frac{2-p}{2}}+\tilde{C} \qquad \forall t \in [a,b]\,, \quad (3.80)$$

and in particular

$$\int_a^t \|u_n(s)\|_X^2 ds+\frac{1}{2}\|w_n(t)\|_H^2 \le C\bigg(\int_a^t \|u_n(s)\|_X^2 dt\bigg)^{\frac{p}{2}}\cdot\bigg(\int_a^t \|w_n(s)\|_H^2 ds\bigg)^{\frac{2-p}{2}}+C \qquad \forall t \in [a,b]\,. \quad (3.81)$$

Assume that $\int_a^t \|u_n(s)\|_X^2 ds \ge 2C$. Then $\frac{1}{2}\int_a^t \|u_n(s)\|_X^2 ds \le \big(\int_a^t \|u_n(s)\|_X^2 ds - C\big)$ and thus by (3.81) we infer

$$\frac{1}{2}\int_a^t \|u_n(s)\|_X^2 ds \le C\bigg(\int_a^t \|u_n(s)\|_X^2 dt\bigg)^{\frac{p}{2}}\cdot\bigg(\int_a^t \|w_n(s)\|_H^2 ds\bigg)^{\frac{2-p}{2}}\,, \quad (3.82)$$

and therefore,

$$\int_a^t \|u_n(s)\|_X^2 ds \le (2C)^{2/(2-p)}\int_a^t \|w_n(s)\|_H^2 ds\,. \quad (3.83)$$

Otherwise we have $\int_a^t \|u_n(s)\|_X^2 ds \le 2C$. Thus there exists a constant $C_1 > 0$ such that in any case we have

$$\int_a^t \|u_n(s)\|_X^2 ds \le C_1\int_a^t \|w_n(s)\|_H^2 ds + C_1 \qquad \forall t \in [a,b] \quad \forall n \in \mathbb{N}\,. \quad (3.84)$$

Plugging it to (3.81), in particular we deduce

$$\|w_n(t)\|_H^2 \le C_2\int_a^t \|w_n(s)\|_H^2 ds + C_2 \qquad \forall t \in [a,b] \quad \forall n \in \mathbb{N}\,, \quad (3.85)$$

where $C_2 > 0$ doesn't depend on n and t. Then

$$\frac{d}{dt}\bigg\{\int_a^t \|w_n(s)\|_H^2 ds\cdot\exp\left(-C_2 t\right)\bigg\} \le C_2\exp\left(-C_2 t\right) \qquad \forall t \in [a,b] \quad \forall n \in \mathbb{N}\,, \quad (3.86)$$

and thus

$$\int_a^t \|w_n(s)\|_H^2 ds \le \exp\left\{C_2(t-a)\right\} - 1 \le \exp\left\{C_2(b-a)\right\} \qquad \forall t \in [a,b] \quad \forall n \in \mathbb{N}\,. \quad (3.87)$$

Therefore, by (3.85) the sequence $\{w_n(t)\}$ is bounded in $L^\infty(a,b;H)$. Moreover, returning to (3.84) and (3.80) we deduce that the sequence $\{u_n\}$ is bounded in $L^q(a,b;X)$. On the other hand since, by (3.67) and the fact that $\{w_n(t)\}$ is bounded in $L^\infty(a,b;H)$ we have $\left\|A_t(S\cdot u_n(t))\right\|_{X^*} \le C\big(\|u_n(t)\|_X^{q-1}+1\big)+\left\|A_t(0)\right\|_{X^*}$, we deduce that $\left\{A_t(S\cdot u_n(t))\right\}$ is bounded in $L^{q^*}(a,b;X^*)$. Moreover, by (3.68), $\left\{\Theta_t\big(u(t)\big)\right\}$ is bounded in $L^{q^*}(a,b;X^*)$. Therefore, by (3.78), using the growth conditions in (3.6) we infer that the sequence $\left\{\frac{dv_n}{dt}(t)\right\}$ is bounded in $L^{q^*}(a,b;X^*)$. So

$$\begin{cases}\{u_n(t)\} & \text{is bounded in } L^q(a,b;X)\,,\\ \left\{\frac{dv_n}{dt}(t)\right\} & \text{is bounded in } L^{q^*}(a,b;X^*)\,,\\ \{w_n(t)\} & \text{is bounded in } L^\infty(a,b;H)\,.\end{cases} \quad (3.88)$$

On the other hand by Corollary 2.1 as in (2.72) for every $a \leq \alpha < \beta \leq b$ and for every $\delta(t) \in C^1\big([a,b];X\big)$ we have

$$\int\limits_\alpha^\beta \bigg\{\Big\langle \delta(t), \frac{dv_n}{dt}(t)\Big\rangle_{X\times X^*} + \Big\langle \frac{d\delta}{dt}(t), v_n(t)\Big\rangle_{X\times X^*}\bigg\}dt = \big\langle T\cdot\delta(\beta), w_n(\beta)\big\rangle_{H\times H} - \big\langle T\cdot\delta(\alpha), w_n(\alpha)\big\rangle_{H\times H}\,. \tag{3.89}$$

However, since S is a compact operator, by (3.88) and Lemma 2.11 we obtain that, up to a subsequence,

$$\begin{cases} u_n(t) \rightharpoonup u(t) \quad \text{weakly in } \; L^q(a,b;X)\,,\\ \frac{dv_n}{dt}(t) \rightharpoonup \zeta(t) \quad \text{weakly in } \; L^{q^*}(a,b;X^*)\,,\\ w_n(t) \rightharpoonup w(t) \quad \text{weakly in } \; L^q(a,b;H)\,,\\ v_n(t) \rightharpoonup v(t) \quad \text{weakly in } \; L^q(a,b;X^*)\,,\\ S\cdot u_n(t) \to S\cdot u(t) \quad \text{strongly in } \; L^q(a,b;Z)\,, \end{cases} \tag{3.90}$$

where $w(t) := T\cdot\big(u(t)\big)$ and $v(t) := \widetilde{T}\cdot\big(w(t)\big)$. In particular, by (3.90) and (3.89) for every $a \leq \alpha < \beta \leq b$ and for every $\delta(t) \in C^1\big([a,b];X\big)$ we have

$$\int\limits_\alpha^\beta \bigg\{\big\langle \delta(t), \zeta(t)\big\rangle_{X\times X^*} + \Big\langle \frac{d\delta}{dt}(t), v(t)\Big\rangle_{X\times X^*}\bigg\}dt = \lim_{n\to+\infty}\Big\{\big\langle T\cdot\delta(\beta), w_n(\beta)\big\rangle_{H\times H} - \big\langle T\cdot\delta(\alpha), w_n(\alpha)\big\rangle_{H\times H}\Big\}\,. \tag{3.91}$$

Thus in particular for every $\delta(t) \in C_c^1\big((a,b);X\big)$ we have

$$\int\limits_a^b \bigg\{\big\langle \delta(t), \zeta(t)\big\rangle_{X\times X^*} + \Big\langle \frac{d\delta}{dt}(t), v(t)\Big\rangle_{X\times X^*}\bigg\}dt = 0\,. \tag{3.92}$$

Therefore, by the definition, $v(t) \in W^{1,q^*}(a,b;X^*)$ and

$$\frac{dv}{dt}(t) = \zeta(t) \qquad \text{for a.e. } \; t \in (a,b)\,. \tag{3.93}$$

Then, as before by Corollary 2.1, $w(t)$ is H-weakly continuous on $[a,b]$ and for every $a \leq \alpha < \beta \leq b$ and for every $\delta(t) \in C^1\big([a,b];X\big)$ we have

$$\int\limits_\alpha^\beta \bigg\{\big\langle \delta(t), \zeta(t)\big\rangle_{X\times X^*} + \Big\langle \frac{d\delta}{dt}(t), v(t)\Big\rangle_{X\times X^*}\bigg\}dt = \big\langle T\cdot\delta(\beta), w(\beta)\big\rangle_{H\times H} - \big\langle T\cdot\delta(\alpha), w(\alpha)\big\rangle_{H\times H}\,. \tag{3.94}$$

Plugging (3.94) into (3.91), for every $a \leq \alpha < \beta \leq b$ and for every $\delta(t) \in C^1\big([a,b];X\big)$ we obtain

$$\lim_{n\to+\infty}\Big\{\big\langle T\cdot\delta(\beta), w_n(\beta)\big\rangle_{H\times H} - \big\langle T\cdot\delta(\alpha), w_n(\alpha)\big\rangle_{H\times H}\Big\} = \big\langle T\cdot\delta(\beta), w(\beta)\big\rangle_{H\times H} - \big\langle T\cdot\delta(\alpha), w(\alpha)\big\rangle_{H\times H}\,. \tag{3.95}$$

In particular for every $h \in X$

$$\lim_{n\to+\infty}\big\langle T\cdot h, w_n(t)\big\rangle_{H\times H} = \big\langle T\cdot h, w(t)\big\rangle_{H\times H} \qquad \forall t \in [a,b]\,. \tag{3.96}$$

Therefore, since by (3.88), $\{w_n\}$ is bounded in $L^\infty(a,b;H)$ and since the image of T is dense in H, using (3.96) we deduce

$$w_n(t) \rightharpoonup w(t) \quad \text{weakly in } \; H \quad \forall t \in [a,b]\,. \tag{3.97}$$

In particular, since $w_n(a) = w_0$ we obtain that $w(a) = w_0$ and so $u(t)$ belongs to $\mathcal{A}_{w_0} = \big\{\psi \in \mathcal{R}_q(a,b) : T\cdot\psi(a) = w_0\big\}$. On the other hand by (3.90), (3.88) and (3.67) we deduce that $A_t\big(S\cdot u_n(t)\big) \to A_t\big(S\cdot u(t)\big)$ strongly in $L^{q^*}(a,b;X^*)$. Moreover, by (3.90) and given properties of Θ_t, we

deduce that $\Theta_t\big(u_n(t)\big) \rightharpoonup \Theta_t\big(u(t)\big)$ weakly in $L^{q^*}(a,b;X^*)$ Therefore, by (3.90), (3.93) and the facts, that we established above, we obtain

$$\begin{cases} u_n(t) \rightharpoonup u(t) \quad \text{weakly in } \ L^q(a,b;X)\,, \\ \frac{dv_n}{dt}(t) \rightharpoonup \frac{dv}{dt}(t) \quad \text{weakly in } \ L^{q^*}(a,b;X^*)\,, \\ \Theta_t\big(u_n(t)\big) \rightharpoonup \Theta_t\big(u(t)\big) \quad \text{weakly in } \ L^{q^*}(a,b;X^*)\,, \\ A_t\big(S\cdot u_n(t)\big) \to A_t\big(S\cdot u(t)\big) \quad \text{strongly in } \ L^{q^*}(a,b;X^*)\,, \\ w_n(b) \rightharpoonup w(b) \quad \text{weakly in } \ H\,, \\ w_n(a) = w(a) = w_0\,. \end{cases} \tag{3.98}$$

On the other hand, by the definition of J_0 in (3.69) and by (3.74) we obtain

$$K = \lim_{n\to\infty} J_0(u_n) = \lim_{n\to\infty} \Bigg(\frac{1}{2}\Big(\|w_n(b)\|_H^2 - \|w_0\|_H^2\Big) + \\ \int\limits_a^b \Bigg\{ \Psi_t\big(u_n(t)\big) + \Psi_t^*\bigg(-\frac{dv_n}{dt}(t) - \Theta_t\big(u_n(t)\big) - A_t\big(S\cdot u_n(t)\big)\bigg) + \\ \Big\langle u_n(t), \Theta_t\big(u_n(t)\big)\Big\rangle_{x\times X^*} + \Big\langle u_n(t), A_t\big(S\cdot u_n(t)\big)\Big\rangle_{X\times X^*}\Bigg\} dt\Bigg)\,. \tag{3.99}$$

However, the functions Ψ_t and Ψ_t^* are convex and Gateux differentiable for every t. Moreover, they satisfy the growth conditions (3.1), (3.6) and the corresponding conditions on their derivatives as stated in (2.13) and (2.14) in Lemma 2.3. Thus $P(x) := \int_a^b \Psi_t(x(t))dt$ and $Q(h) := \int_a^b \Psi_t^*(h(t))dt$ are convex and Gateux differentiable functions on $L^q(a,b;X)$ and $L^{q^*}(a,b;X^*)$ respectively and therefore, $P(x)$ and $Q(h)$ are weakly lower semicontinuous functions on $L^q(a,b;X)$ and $L^{q^*}(a,b;X^*)$ respectively. Moreover, by (3.90) and the given properties of Θ_t, we infer

$$\varliminf_{n\to+\infty} \int_a^b \Big\langle u_n(t), \Theta_t\big(u_n(t)\big)\Big\rangle_{X\times X^*} dt \geq \int_a^b \Big\langle u(t), \Theta_t\big(u(t)\big)\Big\rangle_{X\times X^*} dt\,. \tag{3.100}$$

Therefore, using (3.98), (3.99) and (3.100) we finally obtain

$$\inf_{\theta\in\mathcal{A}_{w_0}} J_0(\theta) = K \geq \frac{1}{2}\Big(\|w(b)\|_H^2 - \|w_0\|_H^2\Big) + \int\limits_a^b \Bigg\{ \Psi_t\big(u(t)\big) + \Psi_t^*\bigg(-\frac{dv}{dt}(t) - \Theta_t\big(u(t)\big) - A_t\big(S\cdot u(t)\big)\bigg) \\ + \Big\langle u(t), \Theta_t\big(u(t)\big)\Big\rangle_{X\times X^*} + \Big\langle u(t), A_t\big(S\cdot u(t)\big)\Big\rangle_{X\times X^*}\Bigg\} dt = J_0(u)\,. \tag{3.101}$$

Thus u is a minimizer to (3.70). Moreover, all the conditions of Theorem 3.1 are satisfied and thus u must satisfy $J_0(u) = 0$ and (3.72). Finally, by Proposition 3.1, the solution to (3.72) is unique. □

A Appendix

Lemma A.1. *Let X, Y and Z be three Banach spaces, such that X is a reflexive space. Furthermore, let $T \in \mathcal{L}(X;Y)$ and $S \in \mathcal{L}(X;Z)$ be bounded linear operators. Moreover assume that S is an injective inclusion (i.e. it satisfies $\ker S = \{0\}$) and T is a compact operator. Then for each $\varepsilon > 0$ there exists some constant $c_\varepsilon > 0$ depending on ε (and on the spaces X, Y, Z and on the operators T, S) such that*

$$\big\|T\cdot h\big\|_Y \leq \varepsilon\big\|h\big\|_X + c_\varepsilon\big\|S\cdot h\big\|_Z \quad \forall h\in X\,. \tag{A.1}$$

Proof. Assume by contradiction that for some $\varepsilon > 0$ such a constant c_e doesn't exist. Then for every natural number $n \in \mathbb{N}$ there exists $h_n \in X$ such that

$$\|T \cdot h_n\|_Y > \varepsilon \|h_n\|_X + n\|S \cdot h_n\|_Z . \tag{A.2}$$

We consider the sequence $\{\xi_n\} \subset X$ defined by the normalization

$$\xi_n := \frac{h_n}{\|h_n\|_X} , \tag{A.3}$$

which satisfy $\|\xi_n\|_X = 1$ and by (A.2),

$$\|T \cdot \xi_n\|_Y > \varepsilon + n\|S \cdot \xi_n\|_Z \qquad \forall n \in \mathbb{N} . \tag{A.4}$$

However, since $\|\xi_n\|_X = 1$, we have $\|T \cdot \xi_n\|_Y \leq \|T\|_{\mathcal{L}(X;Y)}$. So by (A.4) we deduce

$$\|S \cdot \xi_n\|_Z < \frac{1}{n}\|T\|_{\mathcal{L}(X;Y)} \qquad \forall n \in \mathbb{N} .$$

In particular

$$S \cdot \xi_n \to 0 \quad \text{as } \ n \to +\infty \quad \text{strongly in } \ Z . \tag{A.5}$$

On the other hand since $\|\xi_n\|_X = 1$ and since X is a reflexive space, up to a subsequence we must have $\xi_n \rightharpoonup \xi$ weakly in X. Thus $S \cdot \xi_n \rightharpoonup S \cdot \xi$ weakly in Z and then by (A.5) we have $S \cdot \xi = 0$. So since S is an injective operator we deduce that $\xi = 0$ and thus $\xi_n \rightharpoonup 0$ weakly in X. Therefore, since T is a compact operator we have

$$T \cdot \xi_n \to 0 \quad \text{strongly in } \ Y . \tag{A.6}$$

However, returning to (A.4), in particular we deduce

$$\|T \cdot \xi_n\|_Y > \varepsilon \qquad \forall n \in \mathbb{N} , \tag{A.7}$$

which contradicts with (A.6). So we proved (A.1). ∎

Lemma A.2. *Let X be a separable Banach space. Then there exists a separable Hilbert space Y and a bounded linear inclusion operator $S \in \mathcal{L}(Y;X)$ such that S is injective (i.e. $\ker S = \{0\}$), the image of S is dense in X and moreover, S is a compact operator.*

Proof. If X is finite dimensional then X is isomorphic to $\mathbb{R}^k$ for some k and we are done. Otherwise since X is a separable Banach space there exists a countable sequence $\{x_n\}_{n=1}^{+\infty} \subset X$ such that $\|x_n\|_X = 1$ for every n, every finite subsystem of the system $\{x_n\}_{n=1}^{+\infty}$ is linearly independent and the span of $\{x_n\}_{n=1}^{+\infty}$ is dense in X. Set $\bar{Y} := l^2$ where l^2 is a standard separable Hilbert space defined by

$$l^2 := \Big\{\bar{y} = \alpha_n : \mathbb{N} \to \mathbb{R} : \ \sum_{n=1}^{+\infty} \alpha_n^2 < +\infty\Big\} \tag{A.8}$$

with the scalar product

$$\langle \bar{y}_1, \bar{y}_2 \rangle_{\bar{Y} \times \bar{Y}} = \sum_{n=1}^{+\infty} \alpha_n \beta_n \qquad \text{for } \ \bar{y}_1 = \{\alpha_n\}, \ \bar{y}_2 = \{\beta_n\} . \tag{A.9}$$

Next we prove that for every $\bar{y} = \{\alpha_n\}$ there exists a limit in X,

$$x = \lim_{N \to +\infty} \sum_{n=1}^{N} \frac{\alpha_n}{n} \, x_n . \tag{A.10}$$

Indeed since $\|x_n\|_X = 1$ for every $N \in \mathbb{N}$ and $m \in \mathbb{N}$ we have

$$\Big\| \sum_{n=N}^{N+m} \frac{\alpha_n}{n} \, x_n \Big\|_X^2 \leq \Big(\sum_{n=N}^{N+m} \alpha_n^2 \Big) \cdot \Big(\sum_{n=N}^{N+m} \frac{1}{n^2} \Big) \leq \Big(\sum_{n=N}^{+\infty} \alpha_n^2 \Big) \cdot \Big(\sum_{n=N}^{+\infty} \frac{1}{n^2} \Big) \to 0 \quad \text{as } \ N \to +\infty .$$

Thus since X is a Banach space the limit in (A.10) exists. Then define the linear operator $\bar{S} : \bar{Y} \to X$ for every $\bar{y} = \{\alpha_n\} \in \bar{Y}$ by

$$\bar{S} \cdot \bar{y} = \lim_{N\to+\infty} \sum_{n=1}^{N} \frac{\alpha_n}{n}\, x_n\,. \tag{A.11}$$

As before,

$$\left\|\bar{S} \cdot \bar{y}\right\|_X^2 = \left\|\lim_{N\to+\infty} \sum_{n=1}^{N} \frac{\alpha_n}{n}\, x_n\right\|_X^2 \leq \left(\sum_{n=1}^{+\infty} \alpha_n^2\right) \cdot \left(\sum_{n=1}^{+\infty} \frac{1}{n^2}\right) = \left(\sum_{n=1}^{+\infty} \frac{1}{n^2}\right) \cdot \|\bar{y}\|_{\bar{Y}}^2\,. \tag{A.12}$$

Thus $\bar{S}$ is a bounded operator i.e. $\bar{S} \in \mathcal{L}(Y; X)$. Next clearly for every finite linear combination $z = \sum_{n=1}^{N} c_n x_n$ (where $c_n \in \mathbb{R}$) there exists $\bar{y} \in \bar{Y}$ such that $\bar{S} \cdot \bar{y} = z$. So the image of $\bar{S}$ is dense in X. We will prove now that $\bar{S}$ is a compact operator. Indeed let $\bar{y}_n := \{\alpha_j^{(n)}\}_{j=1}^{+\infty} \in \bar{Y}$ be such that $\bar{y}_n \rightharpoonup 0$ weakly in $\bar{Y}$. This means $\lim_{n\to+\infty} \alpha_j^{(n)} = 0$ for every j and $\sum_{j=1}^{+\infty} \left(\alpha_j^{(n)}\right)^2 \leq C$ for some constant $C > 0$. Fix some $\varepsilon > 0$. Then since for every n

$$\left\|\lim_{m\to+\infty} \sum_{j=N}^{N+m} \frac{\alpha_j^{(n)}}{j}\, x_j\right\|_X^2 \leq \left(\sum_{j=1}^{+\infty} \left(\alpha_j^{(n)}\right)^2\right) \cdot \left(\sum_{j=N}^{+\infty} \frac{1}{j^2}\right) \leq C\left(\sum_{j=N}^{+\infty} \frac{1}{j^2}\right) \to 0 \quad \text{as } N \to +\infty\,,$$

there exists N_0 such that

$$\left\|\lim_{m\to+\infty} \sum_{j=N_0}^{N_0+m} \frac{\alpha_j^{(n)}}{j}\, x_j\right\|_X < \frac{\varepsilon}{2} \qquad \forall n \in \mathbb{N}\,. \tag{A.13}$$

On the other hand, since $\lim_{n\to+\infty} \alpha_j^{(n)} = 0$, there exist n_0 such that $|\alpha_j^{(n)}| < \varepsilon/(2N_0)$ for every $n > n_0$ and $1 \leq j \leq N_0$ and thus

$$\left\|\sum_{j=1}^{N_0-1} \frac{\alpha_j^{(n)}}{j}\, x_j\right\|_X < \frac{\varepsilon}{2} \qquad \forall n > n_0\,. \tag{A.14}$$

Plugging (A.14) into (A.13) we deduce that

$$\left\|\bar{S} \cdot \bar{y}_n\right\|_X < \varepsilon \qquad \forall n > n_0\,.$$

Therefore $S \cdot \bar{y}_n \to 0$ strongly in X and so $\bar{S}$ is a compact operator. Finally set $Z := \{\bar{y} \in \bar{Y} :\ \bar{S} \cdot \bar{y} = 0\}$. Then Z is a close subspace of $\bar{Y}$. Next define Y to be the orthogonally dual to Z space

$$Y := \left\{\bar{y} \in \bar{Y} :\ \left\langle \bar{y}, z\right\rangle_{\bar{Y}\times\bar{Y}} = 0 \quad \forall z \in Z\right\}.$$

Then Y is a close subspace of $\bar{Y}$. Therefore Y is a separable Hilbert space by itself. Define $S \in \mathcal{L}(Y; X)$ by $S := \bar{S}\ |_Y$. Then clearly S is injective i.e. $\ker S = \{0\}$. Moreover, if $x = \bar{S} \cdot \bar{y}$ where $\bar{y} \in \bar{Y}$ then we can represent $\bar{y} = z + y$ where $z \in Z$ and $y \in Y$, and since $\bar{S} \cdot z = 0$ we have $x = S \cdot y$. Therefore since the image of $\bar{S}$ is dense in X we deduce that the image of S is also dense in X. Finally S is a compact operator. This completes the proof. □

Lemma A.3. *Let X be a separable Banach space. Then there exists a separable Hilbert space Y and a bounded linear inclusion operator $S \in \mathcal{L}(X; Y)$ such that S is injective (i.e. $\ker S = \{0\}$), the image of S is dense in Y and moreover, S is a compact operator.*

Proof. By the Lindenstrauss's Theorem (see [6]) every separable Banach space is continuously embedded in c_0 where c_0 is a Banach space of real sequences which tend to 0, i.e. it is defined by

$$c_0 := \left\{\alpha_n : \mathbb{N} \to \mathbb{R}\ :\ \lim_{n\to+\infty} a_n = 0\right\}, \qquad \left\|\alpha_n\right\|_{c_0} := \sup_{n\in\mathbb{N}} \left|a_n\right|. \tag{A.15}$$

So there exists an embedding operator $P \in \mathcal{L}(X; c_0)$ which is an injective inclusion (i.e. $\ker P = \{0\}$). Next define $Q \in \mathcal{L}(c_0, l^2)$, where l^2 is the separable Hilbert space defined in (A.8), by the formula

$$Q \cdot \Big(\{\alpha_n\}_{n=1}^{+\infty}\Big) = \left\{\frac{\alpha_n}{n}\right\}_{n=1}^{+\infty} \in l^2 \qquad \forall \{\alpha_n\}_{n=1}^{+\infty} \in c_0\,. \tag{A.16}$$

Then clearly $Q \in \mathcal{L}(c_0, l^2)$ is an injective inclusion. Moreover we will prove now that Q is a compact operator. Indeed for every $j \in \mathbb{N}$ let $h_j := \{\alpha_n^{(j)}\}_{n=1}^{+\infty} \subset c_0$ be such that $h_j \rightharpoonup 0$ weakly in c_0 as $j \to +\infty$. Thus in particular we have

$$\begin{cases} \lim\limits_{j\to+\infty} \alpha_n^{(j)} = 0 \qquad \forall n \in \mathbb{N} \\ |\alpha_n^{(j)}| \le C \qquad \forall n, j \in \mathbb{N}\,, \end{cases} \tag{A.17}$$

for some constant $C > 0$. Then for every $j, m \in \mathbb{N}$ we have

$$\big\|Q \cdot h_j\big\|_{l^2} = \sum_{n=1}^{+\infty} \left(\frac{\alpha_n^{(j)}}{n}\right)^2 = \sum_{n=1}^{m} \left(\frac{\alpha_n^{(j)}}{n}\right)^2 + \sum_{n=m}^{+\infty} \left(\frac{\alpha_n^{(j)}}{n}\right)^2 \le \sum_{n=1}^{m} \left(\frac{\alpha_n^{(j)}}{n}\right)^2 + 4C^2 \sum_{n=m}^{+\infty} \frac{1}{n^2}\,. \tag{A.18}$$

Thus, since $\sum_{n=1}^{+\infty} \frac{1}{n^2} < +\infty$, for every $\varepsilon > 0$ there exists $m = m_\varepsilon \in \mathbb{N}$ such that $4C^2 \sum_{n=m}^{+\infty} \frac{1}{n^2} < \varepsilon$. Therefore, by (A.18) we obtain

$$\big\|Q \cdot h_j\big\|_{l^2} \le \sum_{n=1}^{m} \left(\frac{\alpha_n^{(j)}}{n}\right)^2 + \varepsilon\,. \tag{A.19}$$

Then letting $j \to +\infty$ in (A.19) and using (A.17) we deduce

$$\varlimsup_{j\to+\infty} \big\|Q \cdot h_j\big\|_{l^2} \le \varepsilon\,,$$

and since $\varepsilon > 0$ was arbitrary we finally infer that $Q \cdot h_j \to 0$ strongly in l^2. So we proved that Q is a compact operator. Next define $S \in \mathcal{L}(X; l^2)$ by $S := Q \circ P$, where $P \in \mathcal{L}(X; c_0)$ is an injective embedding. Thus since P and Q are injective, we obtain that S is also an injective embedding of X to l^2. Moreover since Q is a compact operator we obtain that $S \in \mathcal{L}(X; l^2)$ is also a compact operator. Finally let Y be the closure of the image of S in l^2. Then Y is a subspace in l^2 and so the separable Hilbert space by itself. Moreover $S \in (X; Y)$ is an injective compact inclusion of X to Y with dense in Y image. $\square$

Proof of Lemma 2.1. Clearly $\bar{f}(t) \in L^q\big((-\infty, +\infty); X\big)$. Let $H_0 > 0$. Set $A := a - H_0$ and $B := b + H_0$. Then there exists a sequence $\{\bar{f}_n(t)\} \subset C_c^0\big((A, B); X\big)$ such that $\bar{f}_n(t) \to \bar{f}(t)$ in the strong topology of $L^q\big((-\infty, +\infty); X\big)$. Therefore, given $\varepsilon > 0$ there exists $n := n(\varepsilon) \in \mathbb{N}$ such that

$$\left(\int_{\mathbb{R}} \big\|\bar{f}_n(t) - \bar{f}(t)\big\|_X^q dt\right)^{1/q} < \frac{\varepsilon}{3}\,. \tag{A.20}$$

Then for $h \in (-H_0, H_0)$ we have

$$\left(\int_{\mathbb{R}} \big\|\bar{f}(t+h) - \bar{f}(t)\big\|_X^q dt\right)^{1/q} = \left(\int_{\mathbb{R}} \Big\|\big(\bar{f}_n(t) - \bar{f}(t)\big) - \big(\bar{f}_n(t+h) - \bar{f}(t+h)\big) + \big(\bar{f}_n(t+h) - \bar{f}_n(t)\big)\Big\|_X^q dt\right)^{1/q}$$
$$\le \left(\int_{\mathbb{R}} \big\|\bar{f}_n(t) - \bar{f}(t)\big\|_X^q dt\right)^{1/q} + \left(\int_{\mathbb{R}} \big\|\bar{f}_n(t+h) - \bar{f}(t+h)\big\|_X^q dt\right)^{1/q} + \left(\int_{\mathbb{R}} \big\|\bar{f}_n(t+h) - \bar{f}_n(t)\big\|_X^q dt\right)^{1/q}$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \left(\int_A^B \big\|\bar{f}_n(t+h) - \bar{f}_n(t)\big\|_X^q dt\right)^{1/q}. \tag{A.21}$$

However, since $\bar{f}_n(t) \in C_c^0\big((A,B);X\big)$, it is a uniformly continuous function on $\mathbb{R}$. Thus there exists $\delta > 0$ such that for every $h \in (-\delta, \delta)$ and every $t \in \mathbb{R}$ we have $\|\bar{f}_n(t+h) - \bar{f}_n(t)\|_X < \frac{\varepsilon}{3(B-A)^{1/q}}$. Therefore, by (A.21) we deduce that

$$\left(\int_{\mathbb{R}} \big\|\bar{f}(t+h) - \bar{f}(t)\big\|_X^q dt\right)^{1/q} < \varepsilon \quad \forall h \in (-\delta, \delta)\,. \tag{A.22}$$

Thus since $\varepsilon > 0$ was arbitrary small we deuce (2.2).

Next observe that for every $h > 0$,

$$\int_{\mathbb{R}} \left(\frac{1}{h}\int_{-h}^{h} \big\|\bar{f}(t+\tau) - \bar{f}(t)\big\|_X^q d\tau\right) dt = \int_{-1}^{1} \left(\int_{\mathbb{R}} \big\|\bar{f}(t+hs) - \bar{f}(t)\big\|_X^q dt\right) ds\,. \tag{A.23}$$

However, by (2.2), for every $s \in (-1,1)$ we have

$$\lim_{h\to 0} \int_{\mathbb{R}} \big\|\bar{f}(t+hs) - \bar{f}(t)\big\|_X^q dt = 0\,. \tag{A.24}$$

Moreover,

$$\left(\int_{\mathbb{R}} \big\|\bar{f}(t+hs) - \bar{f}(t)\big\|_X^q dt\right)^{1/q} \leq \\ \left(\int_{\mathbb{R}} \big\|\bar{f}(t+hs)\big\|_X^q dt\right)^{1/q} + \left(\int_{\mathbb{R}} \big\|\bar{f}(t)\big\|_X^q dt\right)^{1/q} = 2\left(\int_{\mathbb{R}} \big\|\bar{f}(t)\big\|_X^q dt\right)^{1/q}. \tag{A.25}$$

Therefore, using (A.24) and (A.25), by Dominated Convergence Theorem we deduce that

$$\lim_{h\to 0} \int_{-1}^{1} \left(\int_{\mathbb{R}} \big\|\bar{f}(t+hs) - \bar{f}(t)\big\|_X^q dt\right) ds = 0\,,$$

and thus by (A.23) we deduce (2.3).

Next consider a sequence $\varepsilon_n \to 0^+$ as $n \to +\infty$. Then by (2.3) we have

$$\lim_{n\to+\infty} \int_{\mathbb{R}} \left(\frac{1}{\varepsilon_n}\int_{-\varepsilon_n}^{\varepsilon_n} \big\|\bar{f}(t+\tau) - \bar{f}(t)\big\|_X^q d\tau\right) dt = 0\,. \tag{A.26}$$

Therefore, since every sequence which converges in $L^1(\mathbb{R};\mathbb{R})$ has a subsequence which converges almost everywhere, we finally deduce that up to a subsequence, we have (2.4). □

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